5 The Dynamics of the Shallow-Water Equations

A simple approximation of the dynamic equations of the atmosphere, which can, however, already be used for studying essential questions and methods, is the shallow-water equations. Within this framework we will discuss the concept of the *quasigeostrophic approximation*, and we will introduce *elementary wave types*. Finally we will also discuss the *geostrophic adjustment* process.

5.1 Derivation of the Equations

At the basis of shallow-water dynamics are the following central assumptions:

- We limit ourselves to the range of validity of the primitive equations, as introduced in Sect. 3.4:
  - The aspect ratio between horizontal and vertical scale is large so that the atmosphere can be assumed to be in *hydrostatic equilibrium*, as shown in Sect. 1.6.2.
  - We focus on processes at an altitude above the ground that is much smaller than the Earth's radius, i.e., $z \ll a$.
  - In order to ensure conservation of energy and angular momentum we use the *traditional approximation.*
- The atmosphere is *homogeneous*, i.e., it has constant density. Certainly this is quite strong a limitation since the real atmosphere is compressible. A corresponding generalization will be considered in the following chapter.

Luckily, however, this rather coarse approximation already allows for essential dynamical features which re-appear in the general atmosphere.

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5.1.1 The Momentum Equation

The geometric situation is shown in Fig. 5.1: Above a bottom orography with longitude and latitude dependent height $z_0(\lambda, \phi)$ lies an atmosphere with time-dependent vertical extent $h(\lambda, \phi, t)$. The pressure is in hydrostatic equilibrium with the constant density, i.e.,

$$\frac{\partial p}{\partial z} = -\rho g = \text{const.} \tag{5.1}$$

Integration from the bottom to altitude z yields

$$p(\lambda, \phi, z, t) - p_0(\lambda, \phi, t) = -g\rho\left[z - z_0(\lambda, \phi)\right] \tag{5.2}$$

where p_0 is the bottom pressure. At the upper surface, where $p = 0$ and $z = z_0 + h$, this yields

$$p_0(\lambda, \phi, t) = g\rho h(\lambda, \phi, t) \tag{5.3}$$

Inserting the result into (5.2) we obtain for the pressure the general relationship

$$p(\lambda, \phi, z, t) = g\rho\left[\eta(\lambda, \phi, t) + H - z\right] \tag{5.4}$$

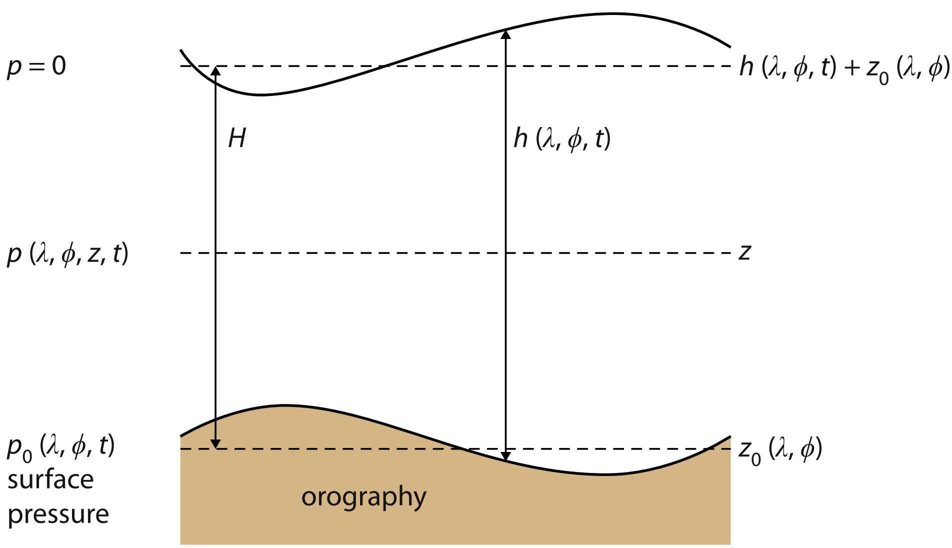

Fig. 5.1 Geometric situation of the shallow-water equations: Above a surface orography of height $z_0(\lambda, \phi)$ lies an atmosphere with vertical extent $h(\lambda, \phi, t)$. The bottom pressure is $p_0(\lambda, \phi, t)$, while the pressure at some altitude z is $p(\lambda, \phi, z, t)$, and at the upper surface $p = 0$. The deviation of the altitude of the upper surface at $z = z_0 + h$ from its equilibrium position $z = H$ is $\eta(\lambda, \phi, t)$

where

$$\boxed{\eta\,(\lambda, \phi, t) = z_0\,(\lambda, \phi) + h\,(\lambda, \phi, t) - H} \tag{5.5}$$

is the the deviation of the vertical position of the altitude of the upper surface from its equilibrium value H. Inserting the pressure from (5.4) into the horizontal-momentum equation (3.48) yields its shallow-water equivalent

$$\frac{D\mathbf{u}}{Dt} + \mathbf{f} \times \mathbf{u} = -g\nabla\eta \tag{5.6}$$

The right-hand side of this equation does not depend on altitude. Thus we can assume that the *horizontal wind* $\mathbf{u}$ *is altitude independent* for all time, leading to the shallow-water momentum equation

$$\boxed{\frac{D\mathbf{u}}{Dt} + \mathbf{f} \times \mathbf{u} = -g\nabla\eta \qquad \frac{\partial}{\partial z} = 0} \tag{5.7}$$

that can be written component-wise

$$\frac{Du}{Dt} - \frac{uv}{a}\tan\phi - fv = -\frac{1}{a\cos\phi}\frac{\partial\eta}{\partial\lambda} \tag{5.8}$$

$$\frac{Dv}{Dt} + \frac{u^2}{a}\tan\phi + fu = -\frac{1}{a}\frac{\partial\eta}{\partial\phi} \tag{5.9}$$

with $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$.

5.1.2 The Continuity Equation

Since density is constant the equation of continuity (3.52) becomes simply $\nabla \cdot \mathbf{v} = 0$, or

$$\frac{\partial w}{\partial z} + \nabla \cdot \mathbf{u} = 0 \tag{5.10}$$

This we integrate from the bottom to the upper surface:

$$w\,(\lambda, \phi, z_0 + h, t) - w\,(\lambda, \phi, z_0, t) + h\nabla \cdot \mathbf{u} = 0 \tag{5.11}$$

The vertical wind, however, is identical with the material derivative of the altitude, i.e.,

$$w\,(\lambda, \phi, z_0 + h, t) = \frac{D}{Dt}\,(z_0 + h) \tag{5.12}$$

$$w\,(\lambda, \phi, z_0, t) = \frac{Dz_0}{Dt} \tag{5.13}$$

so that the vertically integrated continuity equation can be written

$$\boxed{\frac{Dh}{Dt} + h\nabla \cdot \mathbf{u} = 0} \tag{5.14}$$

Alternatively this also yields

$$\boxed{\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0} \tag{5.15}$$

We have thus obtained a closed equation system (5.5), (5.7), and (5.14) or (5.15) for the variables $\mathbf{u}$ and η. In contrast to the general primitive equations the vertical wind and the thermodynamic fields do not explicitly show up anymore. In the remainder of this chapter we will always imply $\partial/\partial z = 0$ without explicit mentioning.

5.1.3 Summary

The shallow-water equations are a simplified model for the atmosphere which already captures essential aspects of atmospheric dynamics.

- Basic assumptions are *hydrostatics*, rather uncritical, but also *homogeneity* of the atmosphere. Finally it is also assumed that the *horizontal wind has no vertical dependence*.
- Prognostic variables are the *horizontal wind* and the *vertical extent of the local atmospheric column*. Horizontal-momentum equation and continuity equation suffice. No thermodynamics is used.

5.2 Conservation Properties

Although several simplifying assumptions have been made in the derivation the shallow-water equation they have similar conservation properties as the general basic equations. Here we show the conservation of energy and potential vorticity.

5.2.1 Energy Conservation

For the demonstration of energy conservation we first multiply the continuity equation (5.14) by gh. One obtains

$$\frac{D}{Dt}\left(g\frac{h^2}{2}\right) + gh^2\nabla \cdot \mathbf{u} = 0 \tag{5.16}$$

Evaluating the material derivative yields

$$\frac{\partial}{\partial t}\left(g\frac{h^2}{2}\right) + \nabla \cdot \left(g\frac{h^2}{2}\mathbf{u}\right) + g\frac{h^2}{2}\nabla \cdot \mathbf{u} = 0 \tag{5.17}$$

Likewise we obtain from the scalar product between the momentum equation (5.7) and $h\mathbf{u}$

$$h\frac{D}{Dt}\frac{|\mathbf{u}|^2}{2} = -gh\mathbf{u}\cdot\nabla\left(z_0+h\right) \tag{5.18}$$

since $\mathbf{u}\cdot\nabla\eta = \mathbf{u}\cdot\nabla\left(z_0+h\right)$. Adding (5.18) to the product of the continuity equation (5.14) with $|\mathbf{u}|^2/2$ leads to

$$\frac{D}{Dt}\left(h\frac{|\mathbf{u}|^2}{2}\right) + \frac{|\mathbf{u}|^2}{2}h\nabla\cdot\mathbf{u} = -gh\mathbf{u}\cdot\nabla\left(z_0+h\right) \tag{5.19}$$

Therein we have

$$gh\mathbf{u}\cdot\nabla\left(z_0+h\right) = g\mathbf{u}\cdot\nabla\frac{h^2}{2} + gh\mathbf{u}\cdot\nabla z_0 \tag{5.20}$$

and further therein, using the time independence of z_0 and the continuity equation (5.14),

$$gh\mathbf{u}\cdot\nabla z_0 = gh\frac{Dz_0}{Dt} = g\frac{D}{Dt}\left(hz_0\right) - gz_0\frac{Dh}{Dt} = g\frac{\partial}{\partial t}\left(hz_0\right) + \nabla\cdot\left(\mathbf{u}ghz_0\right) \tag{5.21}$$

so that

$$gh\mathbf{u}\cdot\nabla\left(z_0+h\right) = g\frac{\partial}{\partial t}\left(hz_0\right) + \nabla\cdot\left(\mathbf{u}ghz_0\right) + g\mathbf{u}\cdot\nabla\frac{h^2}{2} \tag{5.22}$$

This inserted into (5.19) yields

$$\begin{aligned}&\frac{\partial}{\partial t}\left(h\frac{|\mathbf{u}|^2}{2}\right) + \mathbf{u}\cdot\nabla\left(h\frac{|\mathbf{u}|^2}{2}\right) + \frac{|\mathbf{u}|^2}{2}h\nabla\cdot\mathbf{u}\\ &= -\frac{\partial}{\partial t}\left(ghz_0\right) - \nabla\cdot\left(\mathbf{u}ghz_0\right) - g\mathbf{u}\cdot\nabla\frac{h^2}{2}\end{aligned} \tag{5.23}$$

thus

$$\frac{\partial}{\partial t}\left(h\frac{|\mathbf{u}|^2}{2} + ghz_0\right) + \nabla\cdot\left(\mathbf{u}\left[h\frac{|\mathbf{u}|^2}{2} + ghz_0\right]\right) = -g\mathbf{u}\cdot\nabla\frac{h^2}{2} \tag{5.24}$$

Adding (5.17) to (5.24) finally yields the desired conservation equation

$$\boxed{\frac{\partial e}{\partial t} + \nabla\cdot\left[\mathbf{u}\left(e + g\frac{h^2}{2}\right)\right] = 0} \tag{5.25}$$

for the energy density

$$\boxed{e = h\frac{|\mathbf{u}|^2}{2} + gh\left(\frac{h}{2} + z_0\right)} \tag{5.26}$$

of the shallow-water equations. As before in the case of the general basic equations or also the primitive equations the conservation property

$$\frac{dE}{dt} = 0 \tag{5.27}$$

for the energy

$$E = a^2 \int_0^{2\pi} d\lambda \int_{-\pi/2}^{\pi/2} d\phi \cos\phi e \tag{5.28}$$

follows by integration over the total surface of the earth, since the spherical integral over the divergence of the energy flux vanishes.

5.2.2 Potential Vorticity

In many respects the vortex dynamics of the shallow-water equations is similar to the one of the primitive equations in isentropic coordinates, as discussed in the Sects. 4.6.2 and 4.6.3. First we derive the vorticity equation of shallow-water theory. In close analogy to the derivation of the representation (4.105) of the horizontal-momentum equation in isentropic coordinates (4.83) we find that the momentum equation (5.7) can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \frac{\mathbf{u} \cdot \mathbf{u}}{2} + (\zeta + f)\, \mathbf{e}_r \times \mathbf{u} = -g \nabla \eta \tag{5.29}$$

where

$$\zeta = (\nabla \times \mathbf{u})_r = \frac{1}{a \cos\phi} \left[\frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \phi} (\cos\phi\, u) \right] \tag{5.30}$$

is the relative vorticity of the shallow-water equations. Onto this we apply the vertical component of the curl and obtain, again in parallel to the derivation of vorticity equation (4.109) of the primitive equations in isentropic coordinates, the vorticity equation of shallow-water dynamics

$$\boxed{\frac{D}{Dt}(\zeta + f) + (\zeta + f)\, \nabla \cdot \mathbf{u} = 0} \tag{5.31}$$

Finally the continuity equation (5.14) yields

$$\nabla \cdot \mathbf{u} = -\frac{1}{h} \frac{Dh}{Dt} = h \frac{D}{Dt} \left(\frac{1}{h} \right) \tag{5.32}$$

Inserting this into (5.31) leads to the desired conservation equation

$$\boxed{\frac{D\Pi_{SW}}{Dt} = 0} \tag{5.33}$$

for the shallow-water potential vorticity

$$\boxed{\Pi_{SW} = \frac{\zeta + f}{h}} \tag{5.34}$$

which is a materially conserved quantity. The close relationship with the primitive-equation potential vorticity (4.113) is obvious where the height of the atmosphere is to be replaced by the local thickness.

5.2.3 Summary

Horizontal-momentum equation and continuity equation together yield the following *conservation properties*:

- *Energy* as sum of kinetic and potential energy is conserved.
- There is a conserved *potential vorticity* which is structurally identical with the primitive-equation potential vorticity.

5.3 Quasigeostrophic Dynamics

Shallow-water dynamics serves well for the introduction of a useful methodology that we will encounter later on in a more general setting. Focussing on processes with typical scales allows simplifying the dynamical equations so that various aspects can be examined more easily. In quasigeostrophic scaling the focus is on synoptic-scale weather systems in mid-latitudes. In addition we also introduce the much-used β-plane approximation.

5.3.1 The Tangential β-Plane

Since the horizontal scale of synoptic-scale weather systems is $L = \mathcal{O}(10^3\,\text{km})$, and hence $L \ll a$, the curvature of the earth's surface should not play a dominant role in their dynamics. Therefore it is useful for the further considerations to introduce a plane as shown in Fig. 5.2 that is tangential to the earth's surface at a mid latitude reference location $(\lambda, \phi, r) = (\lambda_0, \phi_0, a)$. The *Cartesian* unit vectors of this plane are

$$\mathbf{e}_x = \mathbf{e}_\lambda(\lambda_0, \phi_0, a) \tag{5.35}$$

$$\mathbf{e}_y = \mathbf{e}_\phi(\lambda_0, \phi_0, a) \tag{5.36}$$

The normal unit vector is

$$\mathbf{e}_z = \mathbf{e}_r(\lambda_0, \phi_0, a) \tag{5.37}$$

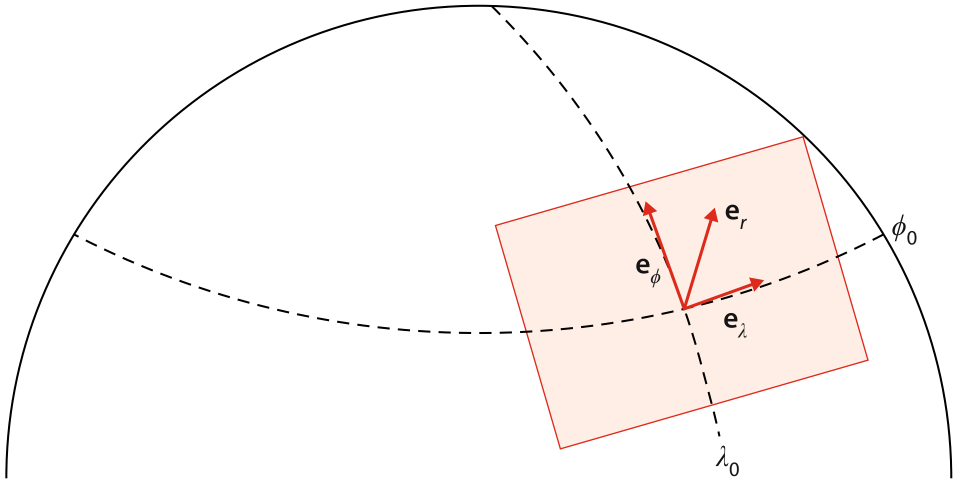

Fig. 5.2 β-plane that is tangential to the earth's surface at longitude λ_0 and latitude ϕ_0. It is spanned by the unit vectors $\mathbf{e}_\lambda$ und $\mathbf{e}_\phi$ at the tangential point. The radial unit vector $\mathbf{e}_r$ at this position is orthogonal to the plane

Consistent with $L \ll a$ we now assume that

$$\lambda - \lambda_0 = \mathcal{O}\left(\frac{L}{a}\right) \ll 1 \tag{5.38}$$

$$\phi - \phi_0 = \mathcal{O}\left(\frac{L}{a}\right) \ll 1 \tag{5.39}$$

so that the corresponding Cartesian coordinates are, as shown in Fig. 5.3,

$$x = a \cos\phi_0 \tan(\lambda - \lambda_0) \approx a \cos\phi_0 (\lambda - \lambda_0) \tag{5.40}$$

$$y = a \tan(\phi - \phi_0) \approx a(\phi - \phi_0) \tag{5.41}$$

We now simply neglect all curvature effects and switch from spherical geometry to the Cartesian geometry of the tangential plane. A more rigorous scale-asymptotic treatment can be found in Chap. 6.

We therefore write the wind as

$$\mathbf{v} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z \tag{5.42}$$

The gradient is

$$\nabla = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} \tag{5.43}$$

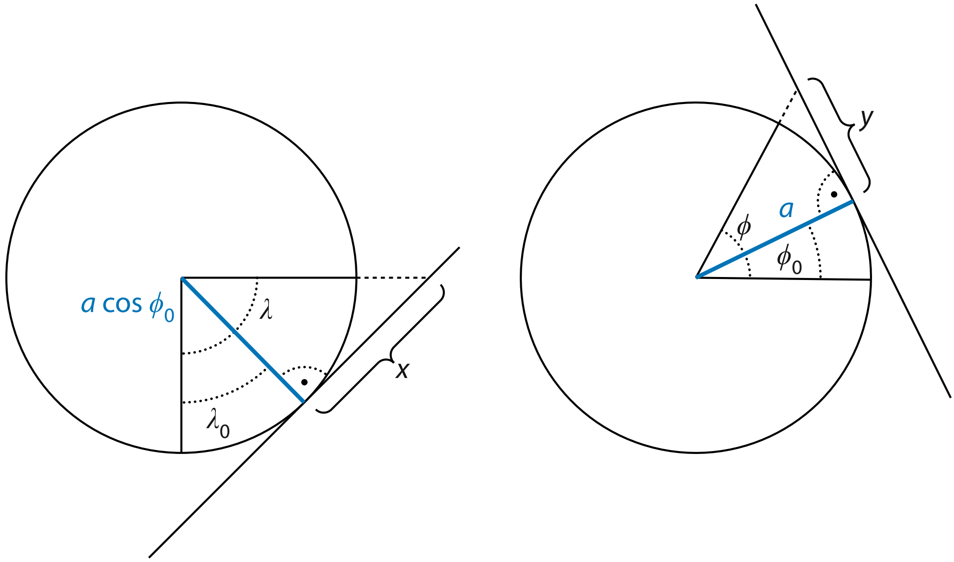

Fig. 5.3 Cartesian coordinates of a β-plane and their relation to longitude and latitude. The left panel shows from a polar perspective the latitude circle at geographic latitude ϕ_0, the right panel the corresponding side view of the meridian at the geographic longitude λ_0

to be used in the material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{5.44}$$

where the the vertical derivatives $\partial/\partial z$ are identified approximately with the radial derivatives. In the special shallow-water context these are all zero. The neglect of all curvature effects simplifies the material derivative of the horizontal wind to

$$\frac{D\mathbf{u}}{Dt} = \frac{Du}{Dt}\mathbf{e}_x + \frac{Dv}{Dt}\mathbf{e}_y \tag{5.45}$$

In the same spirit we approximate everywhere

$$\mathbf{e}_r \approx \mathbf{e}_z \tag{5.46}$$

and hence also

$$\mathbf{f} \approx f\mathbf{e}_z \tag{5.47}$$

where the effect of the latitude dependence of the Coriolis parameter, the so-called *β-effect* is taken into account in an approximate manner. In other words, by (5.39) and (5.41) we have

$$f = 2\Omega \sin\phi \approx 2\Omega \sin\phi_0 + 2\Omega\cos\phi_0(\phi - \phi_0) \approx f_0 + \beta y \tag{5.48}$$

with

$$f_0 = 2\Omega \sin \phi_0 \tag{5.49}$$

$$\beta = \frac{2\Omega}{a} \cos \phi_0 \tag{5.50}$$

Therefore the Coriolis acceleration is approximated by

$$\mathbf{f} \times \mathbf{u} = -fv\mathbf{e}_x + fu\mathbf{e}_y \tag{5.51}$$

where f is to be obtained from (5.48). Moreover, in Cartesian coordinates the pressure-gradient term in the shallow-water momentum equation is

$$g\nabla\eta = g\frac{\partial \eta}{\partial x}\mathbf{e}_x + g\frac{\partial \eta}{\partial y}\mathbf{e}_y \tag{5.52}$$

and the horizontal divergence

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{5.53}$$

so that the shallow-water equations on the β-plane are

$$\frac{D\mathbf{u}}{Dt} + (f_0 + \beta y)\,\mathbf{e}_z \times \mathbf{u} = -\,g\nabla\eta \tag{5.54}$$

$$\frac{Dh}{Dt} + h\nabla \cdot \mathbf{u} = 0 \tag{5.55}$$

where η is still defined via

$$\eta = z_0 + h - H \tag{5.56}$$

Finally we note that on the β-plane vertical relative vorticity is, as generally in Cartesian coordinates,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{5.57}$$

5.3.2 Scaling the Shallow-Water Equations on the β-Plane

Quasigeostrophic theory is a typical example of a simplified theory derived from scale estimates. The principle is illustrated in Fig. 5.4 at the example of a cosine function. Once the order of magnitude F of typical fluctuations of a function $f(x)$ is known as well as the x-scale L within which the function varies typically by this order of magnitude, then an appropriate order-of-magnitude estimate of the derivative is

$$\frac{\partial f}{\partial x} = \mathcal{O}\left(\frac{F}{L}\right) \tag{5.58}$$

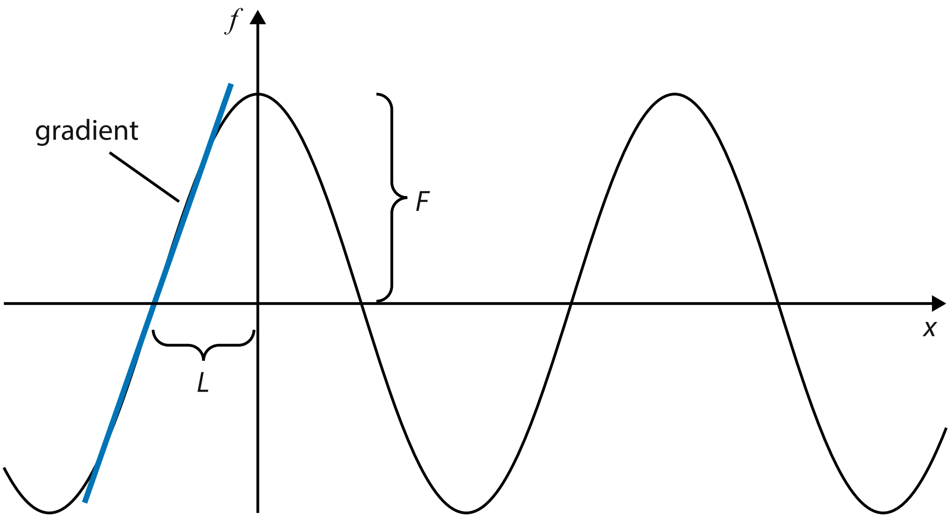

Fig. 5.4 Illustration of a scale estimate at the example of a cosine function. The magnitude of typical gradients can be determined from the ratio between the magnitude δf of typical fluctuations of the function and the typical spatial scale L of these fluctuations

Hence, non-dimensionalizing this function and its independent variable x via

$$f = F\hat{f} \tag{5.59}$$

$$x = L\hat{x} \tag{5.60}$$

leads to

$$\frac{\partial f}{\partial x} = \frac{F}{L}\frac{\partial \hat{f}}{\partial \hat{x}} \tag{5.61}$$

where

$$\frac{\partial \hat{f}}{\partial \hat{x}} = \mathcal{O}(1) \tag{5.62}$$

Along these lines of thought we introduce the following scales for an order-of-magnitude estimate of the derivatives in the equations of motion, oriented at the observation of dominant structures on the weather map: We assume the magnitude of the horizontal-wind fluctuations is U, with 10 m/s being an appropriate numerical value. For the horizontal length scale L we choose the typical extent of pressure and velocity anomalies which can be estimated as 10^3 km. As time scale for the time derivatives we choose the advective time scale $T = L/U$. This corresponds to the time span within which a pressure anomaly passes a geostationary observer. Inserting the numerical values from above one obtains 10^5 s which corresponds to a day. This leads to the non-dimensionalizations

$$(x, y) = L(\hat{x}, \hat{y}) \tag{5.63}$$
$$t = T\hat{t} = \frac{L}{U}\hat{t} \tag{5.64}$$
$$\mathbf{u} = U\hat{\mathbf{u}} \tag{5.65}$$

and thereby for time derivatives and spatial gradients to

$$\frac{\partial}{\partial t} = \frac{1}{T}\frac{\partial}{\partial \hat{t}} \tag{5.66}$$
$$\nabla = \frac{1}{L}\hat{\nabla} \qquad \text{with} \qquad \hat{\nabla} = \mathbf{e}_x\frac{\partial}{\partial \hat{x}} + \mathbf{e}_y\frac{\partial}{\partial \hat{y}} \tag{5.67}$$

where we have used $\partial/\partial z = 0$. Hence the material derivative is

$$\frac{D}{Dt} = \frac{U}{L}\frac{D}{D\hat{t}} \qquad \text{with} \qquad \frac{D}{D\hat{t}} = \frac{\partial}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\nabla} \tag{5.68}$$

and therefore also

$$\frac{D\mathbf{u}}{Dt} = \frac{U^2}{L}\frac{D\hat{\mathbf{u}}}{D\hat{t}} \tag{5.69}$$

For an estimate of the Coriolis term we first note that $f_0 = \mathcal{O}(10^{-4}\mathrm{s}^{-1})$ so that with the given choice for U and L the Rossby number

$$Ro = \frac{U}{f_0 L} \tag{5.70}$$

is $Ro = \mathcal{O}(10^{-1})$, and hence also

$$\boxed{\frac{L}{a} = \mathcal{O}(Ro)} \tag{5.71}$$

With these preparations we obtain for the Coriolis term

$$f = f_0 + \beta y = f_0\hat{f} \tag{5.72}$$

where via (5.48)–(5.50)

$$\hat{f} = \hat{f}_0 + \frac{\beta L}{f_0}\hat{y} \tag{5.73}$$
$$\hat{f}_0 = 1 \tag{5.74}$$
$$\frac{\beta L}{f_0} = \frac{(2\Omega\cos\phi_0/a)\,L}{2\Omega\sin\phi_0} = \frac{L}{a}\cot\phi_0 = Ro\,\hat{\beta} \tag{5.75}$$

with

$$\hat{\beta} = \frac{L/a}{Ro}\cot\phi_0 = \mathcal{O}(1) \tag{5.76}$$

since in mid latitudes

$$\cot \phi_0 = \mathcal{O}(1) \tag{5.77}$$

Therefore one has

$$f_0 + \beta y = f_0 \hat{f} \qquad \text{with} \qquad \hat{f} = \hat{f}_0 + Ro\, \hat{\beta} \hat{y} = \mathcal{O}(1) \tag{5.78}$$

For the pressure gradient we first define a so-far unspecified scale $\mathcal{H}$ of the surface fluctuations, so that

$$\eta = \mathcal{H} \hat{\eta} \tag{5.79}$$

Thereby the pressure-gradient term in the horizontal-momentum equation is

$$g \nabla \eta = \frac{g\mathcal{H}}{L} \hat{\nabla} \hat{\eta} \tag{5.80}$$

Now we insert (5.69), (5.78), and (5.80) into the momentum equation (5.54) and divide by $f_0 U$, with the result

$$Ro \frac{D\hat{\mathbf{u}}}{D\hat{t}} + \hat{f} \mathbf{e}_z \times \hat{\mathbf{u}} = -\frac{g\mathcal{H}}{f_0 U L} \hat{\nabla} \hat{\eta} \tag{5.81}$$

The gain of this non-dimensionalization is clear information on the relative order of magnitude of the various terms in the momentum equation, to be read, respectively, from the pre-factors. To leading order the Coriolis term is $\mathcal{O}(1)$, the wind acceleration $\mathcal{O}(Ro)$, and the pressure-gradient term $\mathcal{O}(g\mathcal{H}/f_0 U L)$. Because

$$\boxed{Ro \ll 1} \tag{5.82}$$

pressure-gradient term and Coriolis term must be of comparable order of magnitude. Otherwise one would find that either the wind or the pressure gradients vanish at the chosen order of magnitude. Hence one obtains for the scale of the surface fluctuations

$$\mathcal{H} = \frac{f_0 U L}{g} \tag{5.83}$$

or

$$\boxed{\mathcal{H} = H\, Ro \frac{L^2}{L_d^2}} \tag{5.84}$$

where

$$\boxed{L_d = \frac{\sqrt{gH}}{f_0}} \tag{5.85}$$

is the *external Rossby deformation radius*. Here H is the mean surface height that can be identified with the mean tropopause height so that $H = 10\,\mathrm{km}$ is a reasonable choice.

Inserting the numbers leads to $L_d = \mathcal{O}(10^3\,\mathrm{km})$.[1] We hence obtain for the momentum equation

$$Ro\frac{D\hat{\mathbf{u}}}{D\hat{t}} + \left(\hat{f}_0 + Ro\hat{\beta}\hat{y}\right)\mathbf{e}_z \times \hat{\mathbf{u}} = -\hat{\nabla}\hat{\eta} \tag{5.86}$$

For a corresponding re-formulation of the continuity equation one needs an estimate $\mathcal{H}_0$ for the order of magnitude of the orography z_0 so that

$$z_0 = \mathcal{H}_0\hat{z}_0 \tag{5.87}$$

A reasonable choice is $\mathcal{H}_0 = 1\,\mathrm{km}$, so that

$$\frac{\mathcal{H}_0}{H} = \mathcal{O}(Ro) \tag{5.88}$$

and hence

$$\frac{\mathcal{H}_0}{H} = Ro\,\hat{h}_0 \qquad \text{with} \qquad \hat{h}_0 = \frac{\mathcal{H}_0/H}{Ro} = \mathcal{O}(1) \tag{5.89}$$

Inserting (5.79), (5.84), and (5.89) into (5.56) then yields for the local vertical atmospheric extent

$$h = H\left(1 + Ro\frac{L^2}{L_d^2}\hat{\eta} - Ro\,\hat{h}_0\hat{z}_0\right) \tag{5.90}$$

This we insert together with (5.68), (5.65), and (5.67) into the continuity equation (5.55) and divide by UH/L, resulting in

$$\frac{D}{D\hat{t}}\left(Ro\frac{L^2}{L_d^2}\hat{\eta} - Ro\,\hat{h}_0\hat{z}_0\right) + \left(1 + Ro\frac{L^2}{L_d^2}\hat{\eta} - Ro\,\hat{h}_0\hat{z}_0\right)\hat{\nabla}\cdot\hat{\mathbf{u}} = 0 \tag{5.91}$$

5.3.3 The Quasigeostrophic Approximation: Derivation by Scale Asymptotics

In the quasigeostrophic approximation of the shallow-water equations we now assume that, consistent with the considerations above,

$$\frac{L}{L_d} \approx \mathcal{O}(1) \tag{5.92}$$

[1] A critical reader will stumble over this in the chapter on the stratified atmosphere. There it will be relevant that a more precise estimate is $L_d \approx 3 \cdot 10^3\,\mathrm{km}$ so that $L^2/L_d^2 = \mathcal{O}(Ro)$. Here we will work with the assumption $L^2/L_d^2 = \mathcal{O}(1)$ which is more consistent with the scalings $L = 3 \cdot 10^3\,\mathrm{km}$ and $U = 30\,\mathrm{m/s}$.

Then in the non-dimensional shallow-water equations (5.86) and (5.91) all factors are $\mathcal{O}(1)$, up to the small parameter Ro. One might first try to simply neglect all terms of $\mathcal{O}(Ro)$ or smaller. The resulting equations, however, are not closed. We therefore also need reasonable estimates for the residuals. To obtain these we expand the solution in terms of the small parameter, i.e., we set

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}_0 + Ro\,\hat{\mathbf{u}}_1 + Ro^2\,\hat{\mathbf{u}}_2 + \ldots \tag{5.93}$$

$$\hat{\eta} = \hat{\eta}_0 + Ro\,\hat{\eta}_1 + Ro^2\,\hat{\eta}_2 + \ldots \tag{5.94}$$

This is to be inserted into the dynamical equations, and then terms of equal power in the small parameter are collected. This way the momentum equation (5.86) becomes

$$Ro\frac{D_0\hat{\mathbf{u}}_0}{D\hat{t}} + \hat{f}_0\mathbf{e}_z \times \hat{\mathbf{u}}_0 + Ro\left(\hat{f}_0\mathbf{e}_z \times \hat{\mathbf{u}}_1 + \hat{\beta}\,\hat{y}\mathbf{e}_z \times \hat{\mathbf{u}}_0\right) = -\hat{\nabla}\hat{\eta}_0 - Ro\,\hat{\nabla}\hat{\eta}_1 + \mathcal{O}\left(Ro^2\right) \tag{5.95}$$

with

$$\frac{D_0}{D\hat{t}} = \frac{\partial}{\partial\hat{t}} + \hat{\mathbf{u}}_0 \cdot \hat{\nabla} \tag{5.96}$$

Collecting all terms of the leading order $\mathcal{O}(1)$ yields the *geostrophic equilibrium*

$$\hat{f}_0\mathbf{e}_z \times \hat{\mathbf{u}}_0 = -\hat{\nabla}\hat{\eta}_0 \tag{5.97}$$

or

$$\hat{u}_0 = -\frac{1}{\hat{f}_0}\frac{\partial\hat{\eta}_0}{\partial\hat{y}} \tag{5.98}$$

$$\hat{v}_0 = \frac{1}{\hat{f}_0}\frac{\partial\hat{\eta}_0}{\partial\hat{x}} \tag{5.99}$$

which can also be written

$$\hat{\mathbf{u}}_0 = \frac{\mathbf{e}_z}{\hat{f}_0} \times \hat{\nabla}\hat{\eta}_0 \tag{5.100}$$

Hence to leading order the flow is non-divergent

$$\hat{\nabla} \cdot \hat{\mathbf{u}}_0 = 0 \tag{5.101}$$

Proceeding likewise with the continuity equation (5.91) leads to

$$\frac{D_0}{D\hat{t}}\left(Ro\,\frac{L^2}{L_d^2}\hat{\eta}_0 - Ro\,\hat{h}_0\hat{z}_0\right) + Ro\,\hat{\nabla} \cdot \hat{\mathbf{u}}_1 = \mathcal{O}(Ro^2) \tag{5.102}$$

where (5.101) has been used. The leading order $\mathcal{O}(Ro)$ therefore is

$$\frac{D_0}{D\hat{t}}\left(\frac{L^2}{L_d^2}\hat{\eta}_0 - \hat{h}_0\hat{z}_0\right) + \hat{\nabla}\cdot\hat{\mathbf{u}}_1 = 0 \tag{5.103}$$

Therein $\nabla\cdot\hat{\mathbf{u}}_1$ can be obtained from the order $\mathcal{O}(Ro)$ of (5.95), component-wise

$$\frac{\partial\hat{u}_0}{\partial\hat{t}} + \hat{\mathbf{u}}_0\cdot\hat{\nabla}\hat{u}_0 - \hat{f}_0\hat{v}_1 - \hat{\beta}\hat{y}\hat{v}_0 = -\frac{\partial\hat{\eta}_1}{\partial\hat{x}} \tag{5.104}$$
$$\frac{\partial\hat{v}_0}{\partial\hat{t}} + \hat{\mathbf{u}}_0\cdot\hat{\nabla}\hat{v}_0 + \hat{f}_0\hat{u}_1 + \hat{\beta}\hat{y}\hat{u}_0 = -\frac{\partial\hat{\eta}_1}{\partial\hat{y}} \tag{5.105}$$

Via $\partial(5.105)/\partial\hat{x} - \partial(5.104)/\partial\hat{y}$ we obtain the *vorticity equation*

$$\frac{\partial\hat{\zeta}_0}{\partial\hat{t}} + \hat{\mathbf{u}}_0\cdot\hat{\nabla}\left(\hat{\zeta}_0 + \hat{\beta}\hat{y}\right) = -\hat{f}_0\hat{\nabla}\cdot\hat{\mathbf{u}}_1 \tag{5.106}$$

where

$$\hat{\zeta}_0 = \frac{\partial\hat{v}_0}{\partial\hat{x}} - \frac{\partial\hat{u}_0}{\partial\hat{y}} \tag{5.107}$$

is the non-dimensional relative vorticity. On the right-hand side of (5.106) we recognize the effect of vortex-tube stretching. Elimination of $\hat{\nabla}\cdot\hat{\mathbf{u}}_1$ from (5.103) and (5.106) finally yields

$$\frac{D_0}{D\hat{t}}\left(\hat{\zeta}_0 + \hat{\beta}\hat{y}\right) = \frac{D_0}{D\hat{t}}\left(\hat{f}_0\frac{L^2}{L_d^2}\hat{\eta}_0 - \hat{f}_0\hat{h}_0\hat{z}_0\right) \tag{5.108}$$

which can also be written as the conservation equation

$$\frac{D_0\hat{q}}{D\hat{t}} = 0 \tag{5.109}$$

for the non-dimensional *quasigeostrophic potential vorticity*

$$\hat{q} = \hat{\zeta}_0 + \hat{f}_0 + \hat{\beta}\hat{y} - \hat{f}_0\frac{L^2}{L_d^2}\hat{\eta}_0 + \hat{f}_0\hat{h}_0\hat{z}_0 \tag{5.110}$$

Because of the leading-order geostrophy of the horizontal flow, finding its expression in (5.98) and (5.99), it is appropriate to define a *streamfunction*

$$\hat{\psi}_0 = \frac{\hat{\eta}_0}{\hat{f}_0} \tag{5.111}$$

so that

$$\hat{u}_0 = -\frac{\partial\hat{\psi}_0}{\partial\hat{y}} \tag{5.112}$$
$$\hat{v}_0 = \frac{\partial\hat{\psi}_0}{\partial\hat{x}} \tag{5.113}$$

or

$$\hat{\mathbf{u}}_0 = \mathbf{e}_z \times \hat{\nabla}\hat{\psi}_0 \tag{5.114}$$

and hence also

$$\hat{\zeta}_0 = \hat{\nabla}^2\hat{\psi}_0 \tag{5.115}$$

Using (5.110)–(5.115) the conservation equation (5.109) finally takes the form

$$\left(\frac{\partial}{\partial \hat{t}} - \frac{\partial \hat{\psi}_0}{\partial \hat{y}}\frac{\partial}{\partial \hat{x}} + \frac{\partial \hat{\psi}_0}{\partial \hat{x}}\frac{\partial}{\partial \hat{y}}\right)\left(\hat{\nabla}^2\hat{\psi}_0 + \hat{f}_0 + \hat{\beta}\hat{y} - \hat{f}_0^2\frac{L^2}{L_d^2}\hat{\psi}_0 + \hat{f}_0\hat{h}_0\hat{z}_0\right) = 0 \tag{5.116}$$

This is a closed prognostic equation for the streamfunction.

The last step to be performed is transforming back to a dimensional representation: We drop the zero index in all variables ($\hat{\mathbf{u}}_0$, $\hat{\eta}_0$ and $\hat{\psi}_0$) and use

$$\hat{t} = \frac{U}{L}t \tag{5.117}$$

$$\left(\hat{x}, \hat{y}\right) = \frac{1}{L}(x, y) \tag{5.118}$$

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{U} \tag{5.119}$$

$$\hat{\nabla} = L\nabla \tag{5.120}$$

Moreover, due to (5.114) the geostrophic wind is

$$\mathbf{u} = U\hat{\mathbf{u}} = U\mathbf{e}_z \times \hat{\nabla}\hat{\psi} = UL\mathbf{e}_z \times \nabla\hat{\psi} \tag{5.121}$$

or

$$\boxed{\mathbf{u} = \mathbf{u}_g = \mathbf{e}_z \times \nabla\psi} \tag{5.122}$$

where the dimensional streamfunction is

$$\psi = UL\hat{\psi} \tag{5.123}$$

that can be rewritten via (5.74), (5.111), (5.79), and (5.83), as

$$\psi = UL\frac{\hat{\eta}}{\hat{f}_0} = UL\frac{\eta}{\mathcal{H}} \tag{5.124}$$

or

$$\boxed{\psi = \frac{g}{f_0}\eta} \tag{5.125}$$

We also have, because of (5.70), (5.118), (5.75), and (5.76),

$$\hat{\beta}\hat{y} = \frac{\beta y}{Ro\, f_0} = \frac{L}{U}\beta y \tag{5.126}$$

(5.74) and (5.123) lead to

$$\hat{f}_0^2 \frac{L^2}{L_d^2} \hat{\psi} = \frac{L}{U} \frac{\psi}{L_d^2} \tag{5.127}$$

and finally also (5.74), (5.87), (5.89), and (5.70) to

$$\hat{f}_0 \hat{h}_0 \hat{z}_0 = \frac{\mathcal{H}_0}{H\,Ro} \frac{z_0}{\mathcal{H}_0} = \frac{f_0 L}{U} \frac{z_0}{H} \tag{5.128}$$

Now we insert (5.117)–(5.120), (5.123), (5.127), and (5.128) into the non-dimensional conservation equation (5.116) and obtain

$$\boxed{\left(\frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\nabla^2 \psi + f_0 + \beta y - \frac{\psi}{L_d^2} + f_0 \frac{z_0}{H}\right) = 0} \tag{5.129}$$

This conservation equation can also be written

$$\boxed{\frac{D_g \pi_{SW}}{Dt} = 0} \tag{5.130}$$

where

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla = \frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} \tag{5.131}$$

is the geostrophic material derivative, and

$$\boxed{\pi_{SW} = \nabla^2 \psi + f_0 + \beta y - \frac{\psi}{L_d^2} + f_0 \frac{z_0}{H}} \tag{5.132}$$

the quasigeostrophic potential vorticity. The enormous gain of this result is that it is no more necessary to separately predict the wind $\mathbf{u}$ and the surface fluctuations η. All boils down to one variable, the streamfunction ψ, which up to a constant factor is identical with η, and from which the winds can be obtained by geostrophy.

5.3.4 The Quasigeostrophic Approximation: Derivation from the Conservation of Shallow-Water Potential Vorticity

An alternative, more heuristic, derivation of quasigeostrophic theory directly employs the scale analysis of the momentum equation and the conservation equation for shallow-water potential vorticity. First we derive from the non-dimensional momentum equation (5.86) *in the limit of small Rossby numbers* to leading order the geostrophic equilibrium

$$\hat{\mathbf{u}} = \mathbf{e}_z \times \hat{\nabla} \hat{\psi} \tag{5.133}$$

where the non-dimensional streamfunction is again given by

$$\hat{\psi} = \frac{\hat{\eta}}{\hat{f}_0} \tag{5.134}$$

Just as above one derives from this (5.122) and (5.125). Thereby, and by (5.57), the relative vorticity becomes

$$\zeta = \nabla^2 \psi \tag{5.135}$$

This is inserted into the shallow-water potential vorticity (5.34), and one obtains, additionally using (5.5) and (5.48),

$$\Pi_{SW} = \frac{\zeta + f}{h} = \frac{\nabla^2 \psi + f_0 + \beta y}{H + \eta - z_0} = \frac{f_0}{H} \frac{1 + \dfrac{\nabla^2 \psi}{f_0} + \dfrac{\beta y}{f_0}}{1 + \dfrac{\eta}{H} - \dfrac{z_0}{H}} \tag{5.136}$$

Herein we estimate the order of magnitude of the respective terms as follows: Due to $\nabla^2 = \hat{\nabla}^2/L^2$ and (5.123) one has

$$\frac{\zeta}{f_0} = Ro\, \hat{\nabla}^2 \hat{\psi} \ll 1 \tag{5.137}$$

(5.118) and (5.75) lead to

$$\frac{\beta y}{f_0} = Ro\, \hat{\beta} \hat{y} \ll 1 \tag{5.138}$$

From (5.118), (5.84) and (5.92) one obtains

$$\frac{\eta}{H} = Ro \frac{L^2}{L_d^2} \hat{\eta} \ll 1 \tag{5.139}$$

and finally from (5.87) and (5.89)

$$\frac{z_0}{H} = Ro\, \hat{h}_0 \hat{z}_0 \ll 1 \tag{5.140}$$

Expanding (5.136) in the small terms in (5.137)–(5.140) yields to leading order

$$\Pi_{SW} \approx \frac{f_0}{H} \left(1 + \frac{\nabla^2 \psi}{f_0} + \frac{\beta y}{f_0} - \frac{\eta}{H} + \frac{z_0}{H} \right) = \frac{f_0}{H} + \frac{\pi_{SW}}{H} \tag{5.141}$$

where π_{SW} is again the quasigeostrophic potential vorticity from (5.132). Inserting this into the conservation equation (5.33) and using geostrophy (5.122) again leads to the quasi-geostrophic conservation equation (5.130).

5.3.5 Summary

By focussing on processes with typical scales the equations can be further simplified, so that various aspects can be understood more easily.

- In the *quasigeostrophic scale estimate* the focus is on *synoptic-scale weather systems in midlatitudes*.
- A useful step is the approximation of the *tangential β-plane*.
- Basic assumptions of quasigeostrophic theory are as follows:
  – The *Rossby number is small*, i.e., the Coriolis force is stronger than the inertial force.
  – The *horizontal scale is small in comparison with the earth's radius*. The ratio between the two scales is of the order of the Rossby number.
  – *Horizontal scale and external Rossby deformation radius* are *of the same order of magnitude*.
  – The *ratio between the orography scale and the mean atmospheric height* is also of the order of the Rossby number and thus *small*.
- The only parameter remaining in the non-dimensional equations is the Rossby number. The scale of the fluctuations of the surface elevation can be determined directly from an analysis of the horizontal-momentum equations, where the pressure gradient must be able to balance the Coriolis force.
- A *Rossby-number expansion of the dynamic variables* yields to leading order the geostrophic equilibrium, so that the horizontal wind can be determined directly from the *atmospheric surface elevations* which take the part of a *streamfunction*. All dynamic variables can be determined from this streamfunction.
- To next approximation one obtains the *conservation equation for quasigeostrophic potential vorticity*. Potential vorticity can be determined from the streamfunction. An inversion is also possible.
- An alternative derivation begins directly from the conservation of general potential vorticity and uses the order-of-magnitude estimates named above.

5.4 Wave Solutions of the Shallow-Water Equations

A useful property of the shallow-water equations is that they admit, already in their comparatively simple formulation, essential waves which contribute to atmospheric variability on various time scales. Both *Rossby waves* and *gravity waves* are solutions of the linear shallow-water equations. These will be discussed in the following. In this context *linearization* is an important tool, by which the dynamics of small perturbations to a steady solution of the equations will be considered. Moreover we will meet within this framework a first application of quasigeostrophic theory.

5.4.1 Perturbation Approach

We examine the shallow-water equations on the β-plane (5.54) and (5.55) without orography. They have the steady solution at rest

$$\mathbf{u} = 0 \tag{5.142}$$

$$\eta = 0 \tag{5.143}$$

Now consider a time-dependent state which deviates from the steady solution only very weakly. In other words, we examine the dynamics of *infinitesimally small perturbations* $\mathbf{u}'$ and η', so that

$$\mathbf{u} = \mathbf{u}' \tag{5.144}$$

$$\eta = \eta' \tag{5.145}$$

Inserting this into the equations yields

$$\frac{\partial \mathbf{u}'}{\partial t} + \left(\mathbf{u}' \cdot \nabla\right) \mathbf{u}' + f \mathbf{e}_z \times \mathbf{u}' = -g \nabla \eta' \tag{5.146}$$

$$\frac{\partial \eta'}{\partial t} + \left(\mathbf{u}' \cdot \nabla\right) \eta' + H \nabla \cdot \mathbf{u}' + \eta' \nabla \cdot \mathbf{u}' = 0 \tag{5.147}$$

with $f = f_o + \beta y$, where we have also used $h = H + \eta$. In the linearization step we neglect all terms which are quadratic in the infinitesimally small perturbation fields, with the result

$$\frac{\partial \mathbf{u}'}{\partial t} + f \mathbf{e}_z \times \mathbf{u}' = -g \nabla \eta' \tag{5.148}$$

$$\frac{\partial \eta'}{\partial t} + H \nabla \cdot \mathbf{u}' = 0 \tag{5.149}$$

For reasons which will become clearer below we change from the component-wise representation of the wind via u and v to a representation in terms of relative vorticity and divergence. First of all, the two components of the momentum equation are as follows:

$$\frac{\partial u'}{\partial t} - f v' = -g \frac{\partial \eta'}{\partial x} \tag{5.150}$$

$$\frac{\partial v'}{\partial t} + f u' = -g \frac{\partial \eta'}{\partial y} \tag{5.151}$$

In an analogous manner as above we obtain from this an equation for relative vorticity via $\partial(5.151)/\partial x - \partial(5.150)/\partial y$. The result is

$$\boxed{\frac{\partial \zeta'}{\partial t} + f \delta' + \beta v' = 0} \tag{5.152}$$

where

$$\zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \tag{5.153}$$

is the linear relative vorticity, and

$$\delta' = \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \tag{5.154}$$

the linear divergence of the wind. Similarly, $\partial(5.150)/\partial x + \partial(5.151)/\partial y$ yields

$$\boxed{\frac{\partial \delta'}{\partial t} - f\zeta' + \beta u' = -g\nabla^2\eta'} \tag{5.155}$$

(5.152) and (5.155) are supplemented by the equation of continuity (5.147) which we here write as

$$\boxed{\frac{\partial \eta'}{\partial t} + H\delta' = 0} \tag{5.156}$$

Now we use the *Helmholtz theorem* for two-dimensional vector fields: Under regular conditions, e.g.,

- either in the presence of periodic boundary conditions in all spatial directions, or if
- ζ' and δ' vanish at infinity,

there is always a velocity potential ϕ and a streamfunction ψ so that

$$\mathbf{u}' = \mathbf{e}_z \times \nabla\psi + \nabla\phi \tag{5.157}$$

thus

$$u' = -\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x} \tag{5.158}$$

$$v' = \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \tag{5.159}$$

By definition this implies

$$\zeta' = \left(\nabla \times \mathbf{u}'\right)_z = \nabla^2\psi \tag{5.160}$$

$$\delta' = \nabla \cdot \mathbf{u}' = \nabla^2\phi \tag{5.161}$$

The meaning of streamfunction and velocity potential is illustrated in Fig. 5.5. They describe respectively, the non-divergent and the irrotational part of the flow. (5.157)–(5.161) inserted into (5.152), (5.155) and (5.156) yield

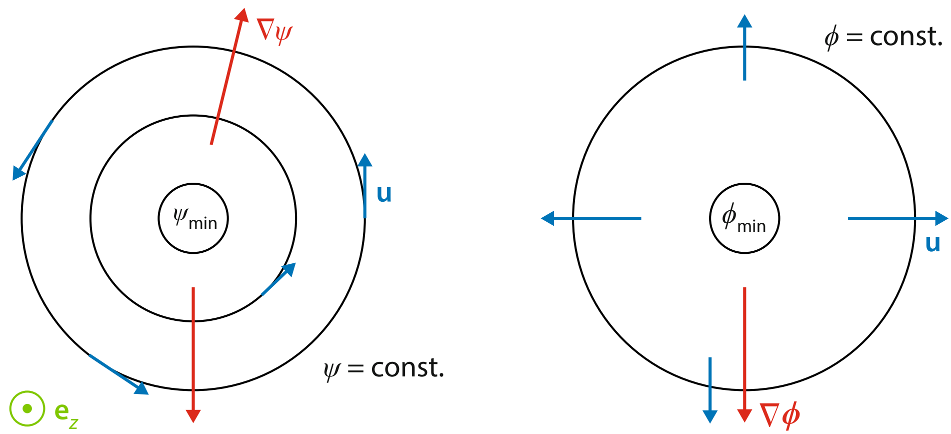

Fig. 5.5 Illustration of the meaning of of streamfunction and velocity potential. A flow with only nonzero streamfunction part follows the isolines of the streamfunction (left panel). This flow has no sinks and sources and thus is non-divergent. In the opposite case of only the velocity potential being nonzero the flow is everywhere orthogonal to the isolines of the the velocity potential (right). This flow is non-rotational and thus has zero relative vorticity

$$\frac{\partial \nabla^2 \psi}{\partial t} + f \nabla^2 \phi + \beta \left(\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \right) = 0 \tag{5.162}$$

$$\frac{\partial \nabla^2 \phi}{\partial t} - f \nabla^2 \psi + \beta \left(-\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x} \right) = -g \nabla^2 \eta' \tag{5.163}$$

$$\frac{\partial \eta'}{\partial t} + H \nabla^2 \phi = 0 \tag{5.164}$$

These equations shall be examined more closely in the following.

5.4.2 Waves on the f-Plane

First consider the approximation of an f*-plane* on which the latitudinal dependence of the Coriolis parameter, the β effect, is neglected. One then has $\beta = 0$ and $f = f_0$, and (5.162)–(5.164) become

$$\frac{\partial \nabla^2 \psi}{\partial t} + f_0 \nabla^2 \phi = 0 \tag{5.165}$$

$$\frac{\partial \nabla^2 \phi}{\partial t} - f_0 \nabla^2 \psi = -g \nabla^2 \eta' \tag{5.166}$$

$$\frac{\partial \eta'}{\partial t} + H \nabla^2 \phi = 0 \tag{5.167}$$

The Non-Rotational Case

It is instructive to further consider the case in which the earth's rotation is neglected as a whole, i.e., with $f_0 = 0$. Then ψ is completely decoupled from ϕ and η'. One can thus find both solutions to which only ψ contributes (non-divergent flow) and such to which only ϕ and η' contribute (irrotational flow). The first is the non-divergent vortical mode, while the latter are irrotational gravity waves. We first consider those.

External Gravity Waves, Phase and Group Velocity The development of ϕ and η' is described by (5.166) and (5.167) with $f_0 = 0$, i.e.,

$$\frac{\partial \nabla^2 \phi}{\partial t} = -g\nabla^2 \eta' \tag{5.168}$$

$$\frac{\partial \eta'}{\partial t} + H\nabla^2 \phi = 0 \tag{5.169}$$

The time derivative of (5.169), followed by application of (5.168), leads to

$$\frac{\partial^2 \eta'}{\partial t^2} - gH\nabla^2 \eta' = 0 \tag{5.170}$$

Here it is appropriate to choose a representation of η' as Fourier integral

$$\eta' = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} d\omega e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tilde{\eta}(\mathbf{k}, \omega) \tag{5.171}$$

where $\mathbf{k} = k\mathbf{e}_x + l\mathbf{e}_y$ is the two-dimensional wave vector, with components k and l in x- and y-direction. Some properties of Fourier integrals are summarized in Appendix 11.5.1. Fourier transformation of (5.170) in space and time yields

$$\left(-\omega^2 + gHK^2\right) \tilde{\eta}(\mathbf{k}, \omega) = 0 \tag{5.172}$$

where $K^2 = k^2 + l^2$ is the squared norm of the wave vector. Obviously the transforms $\tilde{\eta}(\mathbf{k}, \omega)$ need only then not vanish if the bracket is zero. This yields the *dispersion relation for external gravity waves without rotation*

$$\boxed{\omega = \omega_\pm(\mathbf{k}) = \pm\sqrt{gH}K = \pm\sqrt{gH}\sqrt{k^2 + l^2}} \tag{5.173}$$

For each wave vector there are two possibilities. The fact that only on these branches the Fourier transform may be nonzero can be also be expressed via

$$\tilde{\eta}(\mathbf{k}, \omega) = a_+(\mathbf{k})\delta[\omega - \omega_+(\mathbf{k})] + a_-(\mathbf{k})\delta[\omega - \omega_-(\mathbf{k})] \tag{5.174}$$

with arbitrary amplitudes $a_+(\mathbf{k})$ and $a_-(\mathbf{k})$, so that

$$\eta'(t,\mathbf{x}) = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \left\{a_+(\mathbf{k})e^{i[\mathbf{k}\cdot\mathbf{x}-\omega_+(\mathbf{k})t]} + a_-(\mathbf{k})e^{i[\mathbf{k}\cdot\mathbf{x}-\omega_-(\mathbf{k})t]}\right\} \tag{5.175}$$

η' thus is a superposition of gravity waves with both possible dispersion relations (5.173).

The *phase velocity* of each of both waves, parallel or antiparallel to the wave vector, respectively, is

$$\boxed{\mathbf{c}_\pm(\mathbf{k}) = \frac{\omega_\pm}{K}\frac{\mathbf{k}}{K}} \tag{5.176}$$

so that

$$\omega_\pm = \mathbf{c}_\pm \cdot \mathbf{k} \tag{5.177}$$

The meaning of the phase can be understood by considering the corresponding *wave phase*

$$\alpha_\pm(\mathbf{k},\mathbf{x},t) = \mathbf{k}\cdot\mathbf{x} - \omega_\pm(\mathbf{k})\,t = \mathbf{k}\cdot\left[\mathbf{x} - \mathbf{c}_\pm(\mathbf{k})\,t\right] \tag{5.178}$$

so that the two contributions to the Fourier integral are, respectively,

$$a_\pm(\mathbf{k})\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_\pm t)} = a_\pm(\mathbf{k})\,e^{i\alpha_\pm(\mathbf{k},\mathbf{x},t)} \tag{5.179}$$

Along lines of constant phase the respective wave contributions are constant. They are parallel to the wave's ridges and troughs. The question as to which is the location $\mathbf{x}(t)$ where the wave phase keeps its value, i.e., $\alpha(\mathbf{x},t)$ =const., is answered via

$$\frac{d}{dt}\alpha_\pm[\mathbf{x}(t),t] = 0 \tag{5.180}$$

by

$$\frac{d\mathbf{x}}{dt} = \mathbf{c}_\pm \tag{5.181}$$

The phase velocity therefore is the velocity parallel or anti-parallel to the wave vector by which lines of constant phase propagate. This is illustrated in Fig. 5.6.

Finally we also introduce the concept of *group velocity*. For this purpose we consider for each of both gravity-wave solutions a wave packet with contributions only from wave vectors in the vicinity of the central wave vector $\mathbf{k} = \mathbf{k}_0$:

$$\eta'(\mathbf{x},t) = \sum_{\beta=\pm}\int_{k_0-\Delta k}^{k_0+\Delta k} dk \int_{l_0-\Delta l}^{l_0+\Delta l} dl\, a_\beta(\mathbf{k})e^{i[\mathbf{k}\cdot\mathbf{x}-\omega_\beta(\mathbf{k})t]} \tag{5.182}$$

Here $\Delta\mathbf{k}$ is small. The sum of the packets can also be written as

$$\eta'(\mathbf{x},t) = \sum_{\beta=\pm} e^{i[\mathbf{k}_0\cdot\mathbf{x}-\omega_\beta(\mathbf{k}_0)t]}A_\beta(\mathbf{x},t) \tag{5.183}$$

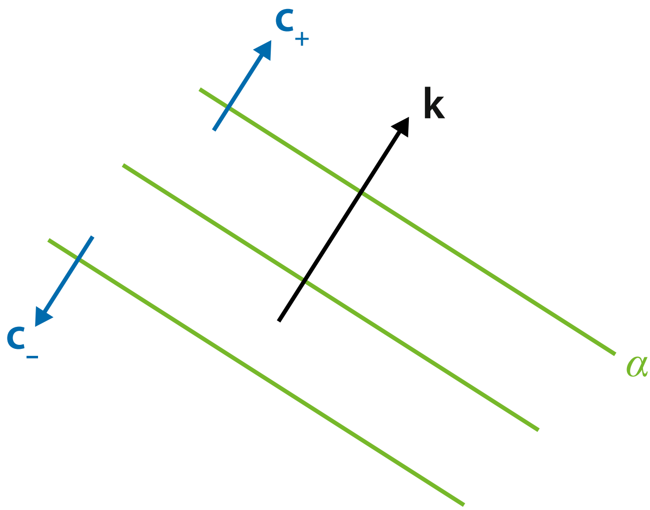

Fig. 5.6 Illustration of the meaning of a phase velocity, at the example of gravity waves without rotation. The phase velocity $\mathbf{c}_{\pm}$ is parallel or anti-parallel to the wave vector $\mathbf{k}$ which again is perpendicular to the lines of constant phase α. It is the velocity by which these lines propagate

where

$$A_\beta(\mathbf{x}, t) = \int_{-\Delta k}^{+\Delta k} dk' \int_{-\Delta l}^{+\Delta l} dl'\, a_\beta(\mathbf{k}_0 + \mathbf{k}')e^{i\{\mathbf{k}'\cdot\mathbf{x} - [\omega_\beta(\mathbf{k}_0+\mathbf{k}') - \omega_\beta(\mathbf{k}_0)]t\}} \tag{5.184}$$

are the respective *envelopes*. Since the only important contributions to the integral come from wave vectors only slightly different from $\mathbf{k}_0$ the frequency can be expanded about this central wave vector, i.e.,

$$\omega_\beta(\mathbf{k}_0 + \mathbf{k}') \approx \omega_\beta(\mathbf{k}_0) + \mathbf{c}_{g,\beta}\,(\mathbf{k}_0) \cdot \mathbf{k}' \tag{5.185}$$

Here the *group velocity*

$$\boxed{\mathbf{c}_{g,\beta}\,(\mathbf{k}_0) = \nabla_{\mathbf{k}}\omega_\beta\big|_{\mathbf{k}_0}} \tag{5.186}$$

is the gradient of the frequency in wave-vector space. Finally (5.184) becomes with (5.185)

$$\begin{aligned} A_\beta(\mathbf{x}, t) &= \int_{-\Delta k}^{+\Delta k} dk' \int_{-\Delta l}^{+\Delta l} dl' a_\beta(\mathbf{k}_0 + \mathbf{k}')e^{i\mathbf{k}'\cdot[\mathbf{x} - \mathbf{c}_{g,\beta}(\mathbf{k}_0)t]} \\ &= A_\beta(\mathbf{x} - \mathbf{c}_{g,\pm}t, 0) \end{aligned} \tag{5.187}$$

We thus see that the envelopes move with their respective group velocities. The general situation is shown in Fig. 5.7. Note that in general phase and group velocity need not coincide! Here, however, this is the case, since the dispersion relation (5.173) leads to

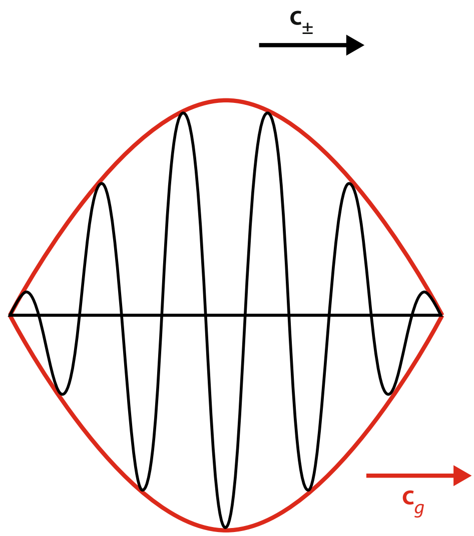

Fig. 5.7 Illustration of the concept of group velocity. A wave packet is a wave train which is modulated by an envelope. While the wave train, i.e., its phase, propagates at the phase velocity, the envelope moves at the group velocity

$$\begin{aligned}\mathbf{c}_{g,\pm} &= \pm\nabla_{\mathbf{k}}\left(\sqrt{gH}\sqrt{k^2+l^2}\right) = \pm\sqrt{gH}\frac{\mathbf{k}}{\sqrt{k^2+l^2}} \\ &= \pm\frac{\sqrt{gH}\sqrt{k^2+l^2}}{\sqrt{k^2+l^2}}\frac{\mathbf{k}}{\sqrt{k^2+l^2}} = \mathbf{c}_\pm \end{aligned} \tag{5.188}$$

In this special case the gravity waves are *non-dispersive*.

The Vortical Mode The differential equation (5.165) for the vortical mode simply is

$$\frac{\partial \nabla^2 \psi}{\partial t} = 0 \tag{5.189}$$

Via a Fourier ansatz for ψ one finds $\omega K^2 = 0$, and thus the trivial dispersion relation

$$\omega = 0 \tag{5.190}$$

Clearly this steady mode has vanishing group and phase velocity.

With Rotation

In the case $f_0 \neq 0$ the three prognostic fields must be treated together. Again, however, we analyze them in Fourier space, i.e., we write them as

$$\begin{pmatrix}\psi\\ \phi\\ \eta'\end{pmatrix}(\mathbf{x},t) = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} d\omega e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \begin{pmatrix}\tilde{\psi}\\ \tilde{\phi}\\ \tilde{\eta}\end{pmatrix}(\mathbf{k},\omega) \tag{5.191}$$

Fourier transformation of (5.165)–(5.167) yields, by the rules in Appendix E,

$$i\omega K^2\tilde{\psi} - f_0 K^2\tilde{\phi} = 0 \tag{5.192}$$

$$i\omega K^2\tilde{\phi} + f_0 K^2\tilde{\psi} - gK^2\tilde{\eta} = 0 \tag{5.193}$$

$$-i\omega\tilde{\eta} - HK^2\tilde{\phi} = 0 \tag{5.194}$$

or

$$B\begin{pmatrix}\tilde{\psi}\\ \tilde{\phi}\\ \tilde{\eta}\end{pmatrix} = 0 \tag{5.195}$$

where the coefficient matrix is

$$B(\mathbf{k}, \omega) = \begin{pmatrix} i\omega K^2 & -f_0 K^2 & 0 \\ f_0 K^2 & i\omega K^2 & -gK^2 \\ 0 & -HK^2 & -i\omega \end{pmatrix} \tag{5.196}$$

For (5.195) to also have non-trivial solutions $\left(\tilde{\psi}, \tilde{\phi}, \tilde{\eta}\right) \neq 0$, the matrix must be singular, i.e., its determinant must vanish

$$\det B = 0 \tag{5.197}$$

One obtains

$$\omega\left(\omega^2 - f_0^2 - gHK^2\right) = 0 \tag{5.198}$$

This leads to two different dispersion relations, for external gravity waves and the geostrophic mode. Both shall be discussed in the following.

Geostrophic Flow One possible solution of (5.198) is

$$\boxed{\omega = 0} \tag{5.199}$$

The corresponding wave solution thus is *steady*, i.e., it does not depend on time. Its structure can best be recognized by referring to the linear equations (5.148) and (5.149), with $f = f_0$, and setting the time derivatives there to zero. One obtains

$$f_0\mathbf{e}_z \times \mathbf{u}' = -g\nabla\eta' \tag{5.200}$$

$$\nabla\cdot\mathbf{u}' = 0 \tag{5.201}$$

The flow is thus non-divergent and in geostrophic equilibrium.

External Inertia-Gravity Waves The two other possible solutions of (5.198) are, using (5.85),

$$\boxed{\omega = \pm\sqrt{f_0^2 + gHK^2} = \pm f_0\sqrt{1 + K^2L_d^2}} \tag{5.202}$$

The structure of the corresponding wave can be obtained from (5.192) and (5.194):

$$\tilde{\eta} = i\frac{HK^2}{\omega}\tilde{\phi} \tag{5.203}$$

$$\tilde{\psi} = -i\frac{f_0}{\omega}\tilde{\phi} \tag{5.204}$$

The surface fluctuations and the streamfunction are opposite in phase, while they are in quadrature to the velocity potential. The latter means that zeros of one quantity coincide with extrema of the other, and vice versa. The gravity-wave dynamics can be illuminated further in the limit of either small or large wavelengths:

Large Wavelengths ($K^2L_d^2 \ll 1$): External Inertia-Gravity Waves In this limit

$$\omega \approx \pm f_0 \tag{5.205}$$

By way of (5.203) and (5.204) one finds that

$$gK^2\tilde{\eta} = \frac{igHK^4}{\omega}\tilde{\phi} = -\frac{gH}{f_0}K^4\tilde{\psi} = -f_0K^4L_d^2\tilde{\psi} \tag{5.206}$$

so that

$$|gK^2\tilde{\eta}| \ll |f_0K^2\tilde{\psi}| \tag{5.207}$$

In the transformed divergence equation (5.193) the contribution from the surface fluctuations can therefore be neglected, and hence also in the divergence equation (5.166). The system of the thus approximated equations is, together with the vorticity equation (5.165),

$$\frac{\partial}{\partial t}\nabla^2\psi + f_0\nabla^2\phi = 0 \tag{5.208}$$

$$\frac{\partial}{\partial t}\nabla^2\phi - f_0\nabla^2\Psi \approx 0 \tag{5.209}$$

These equations describe inertia waves which can also be obtained from the momentum equation

$$\frac{\partial \mathbf{u}'}{\partial t} + f_0\mathbf{e}_z \times \mathbf{u}' = 0 \tag{5.210}$$

without pressure-gradient contribution. It is easily checked that its curl and divergence together yield the equation system (5.208) and (5.209).

Small Wavelengths ($K^2L_d^2 \gg 1$): High-Frequency Gravity Waves In complete analogy to the above one finds in this case that in the divergence equation now the streamfunction contribution can be neglected. It is then coupled to the continuity equation (5.167). In total the approximated equation system is

$$\frac{\partial \nabla^2 \phi}{\partial t} \approx -g\nabla^2 \eta' \tag{5.211}$$

$$\frac{\partial \eta'}{\partial t} + H\nabla^2 \phi = 0 \tag{5.212}$$

These are, however, exactly the basic equations of external gravity waves in the non-rotating system.

5.4.3 Waves on the β Plane: Quasigeostrophic Rossby Waves

With a latitude-dependent Coriolis parameter $f = f_0 + \beta y$ the Fourier approach of the chapter above is not directly applicable anymore. A Fourier transform of the momentum equations in y-direction no more generates contributions from fields at only a single considered wavenumber l. Thus the transformed equations become a complex system coupling all wavenumbers in y-direction. As a result the oscillating solutions turn out to have a much more complex latitudinal structure than that of a monochromatic wave. A corresponding treatment is mathematically possible, but beyond the framework of this course. Luckily, however, now the advantages of the quasigeostrophic approximation become apparent for the first time.

Rossby Waves: Dispersion Relation, Phase and Group Velocity

Within the framework of synoptic scaling, $Ro \ll 1$, the flow structures are geostrophic, thus nearly non-divergent, and their time dependence is determined by the conservation equation (5.129) of quasigeostrophic potential vorticity. Since we are here not interested in orographic effects it is

$$\left(\frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\nabla^2 \psi + \beta y - \frac{\psi}{L_d^2}\right) = 0 \tag{5.213}$$

We now examine which wave solutions can be obtained from this approximation. The state at rest about which we had linearized above is given by, without limitations of generality,

$$\psi = 0 \tag{5.214}$$

One easily convinces oneself that it solves (5.213). Now consider an infinitesimally small perturbation of this state at rest, i.e., assume

$$\psi = \psi' \tag{5.215}$$

and neglect in (5.213) all terms nonlinear in ψ'. The result is

$$\frac{\partial}{\partial t}\nabla^2\psi' + \beta\frac{\partial\psi'}{\partial x} - \frac{1}{L_d^2}\frac{\partial\psi'}{\partial t} = 0 \tag{5.216}$$

Since both β and L_d are constants, this equation can easily be Fourier transformed. One obtains

$$\left(i\omega K^2 + i\beta k + \frac{i\omega}{L_d^2}\right)\tilde{\psi} = 0 \tag{5.217}$$

Hence follows the dispersion relation

$$\boxed{\omega = -\frac{\beta k}{K^2 + \frac{1}{L_d^2}}} \tag{5.218}$$

It describes the spatio-temporal behavior of quasigeostrophic *Rossby waves*. Before we discuss their dynamics we note the following:

- Rossby waves are the basic structures of daily weather. They describe the typical chain of high- and low-pressure systems traversing the globe at midlatitudes. Examples can be seen in Figs. 6.2 and 6.5.
- Quasigeostrophic dynamics does not yield gravity waves. It is a *filtered* dynamics. This results from the assumption of small Rossby numbers, which is equivalent to $f_0T \gg 1$, where T is the time scale of the considered phenomena. Since the gravity-wave frequency is always larger than f_0, their period T satisfies $f_0T < 2\pi$, so that they do not satisfy the basic assumptions of the quasigeostrophic approximation. Since the computational demand rises considerably when gravity waves and also the process, discussed below, of geostrophic adjustment of an arbitrary initial state to its geostrophic equilibrium, via the radiation of gravity waves, are to be simulated, first-generation weather-forecast models have exploited the nonexistence of gravity waves in quasigeostrophic dynamics and based their formulation on the quasigeostrophic approximation.
- The zonal phase-velocity component

$$c_x = \frac{\omega}{K}\frac{k}{K} = -\frac{\beta k^2}{K^2\left(K^2 + \frac{1}{L_d^2}\right)} \tag{5.219}$$

  of a Rossby wave is negative so that the phase propagation is westward. This is not in contradiction with the typical eastward propagation of synoptic-scale weather systems in midlatitudes. The latter results from the strong eastwards-directed mean flow on which the pressure anomalies propagate. Linearizing (5.213) about a streamfunction

$$\psi = -Uy \tag{5.220}$$

i.e., a basic flow with zonal velocity U, one obtains the dispersion relation

$$\omega = \frac{UK^2 - \beta}{K^2 + \frac{1}{L_d^2}} k \tag{5.221}$$

For $U > \beta/K^2$ the corresponding waves indeed propagate eastwards. A stationary wave with frequency $\omega = 0$ is given when

$$K = \sqrt{\beta/U} \tag{5.222}$$

Such waves are excited by flow over mountains or the land–sea contrast in atmospheric heating. Clearly this is only possible if $U > 0$. This explains the difference in the result of westward or eastward flow over a mountain ridge as already discussed in section 4.6.4.

- Rossby waves are highly dispersive. At $U = 0$ the zonal component of their group velocity is, e.g.,

$$c_{g,x} = -\frac{\beta\left(K^2 + \frac{1}{L_d^2} - 2k^2\right)}{\left(K^2 + \frac{1}{L_d^2}\right)^2} \tag{5.223}$$

Not even in the sign this necessarily agrees with (5.219).

- On the f-plane, i.e., at $\beta = 0$, Rossby waves coincide with the geostrophic solution $\omega = 0$.
- At least for small-scale gravity waves the latitude dependence of the Coriolis parameter should not play a major role. A more general treatment shows that even on the β-plane gravity waves with the properties discussed above exist. A summary of all important dispersion relations is given in Fig. 5.8.

Interpretation of Rossby Waves

In the following the Rossby-wave dynamics shall be discussed in somewhat more detail. We begin with the linear conservation equation (5.213) for quasigeostrophic potential vorticity. We here give it together with the conservation equation (5.33) for general potential vorticity so as to highlight the correspondence between respective terms:

$$\begin{array}{ccccccc} & \dfrac{\partial \nabla^2 \psi'}{\partial t} & + \beta \dfrac{\partial \psi'}{\partial x} & - & \dfrac{1}{L_d^2}\dfrac{\partial \psi'}{\partial t} & & = 0 \\ & (1) & (2) & & (3) & & \\ \dfrac{D}{Dt}\,[(& \zeta & + \quad f &)\,/\,(& \eta & +H)] & = 0 \end{array}$$

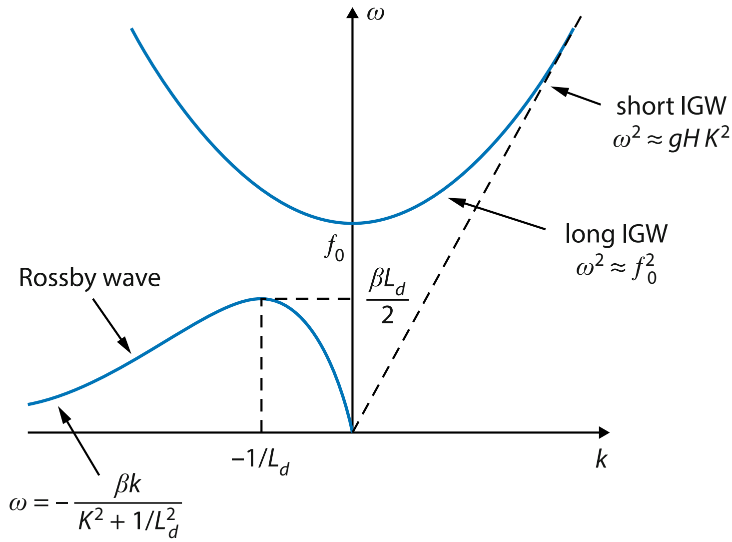

Fig. 5.8 Summary of the dispersion relations of the basic wave types of the shallow-water equations: Rossby waves and inertia-gravity waves (IGW). At large wavelengths the latter become inertia waves, while they are high-frequency gravity waves at short wavelengths

The separate contributions are (1) the local change of relative vorticity, (2) the advection of planetary vorticity, and (3) vortex-tube stretching.

Short Wavelengths ($K^2 \gg 1/L_d^2$): In this case vortex-tube stretching (3) can be neglected in comparison with the local change of relative vorticity (1). One has approximately

$$\frac{\partial \nabla^2 \psi'}{\partial t} + \beta \frac{\partial \psi'}{\partial x} \approx 0 \tag{5.224}$$

which corresponds to the linear approximation of

$$\frac{D}{Dt}(\zeta + f) = 0 \tag{5.225}$$

This means that in such waves the absolute vorticity, consisting of relative and planetary vorticity, is conserved. A Rossby wave can thus be viewed as a chain of streamfunction anomalies with positive and negative sign. Now consider such an anomaly as in Fig. 5.9. There we see a top view of a positive relative-vorticity anomaly on the northern hemisphere. This corresponds to a negative streamfunction anomaly, thus a low-pressure system. On its eastern side air masses move northwards. Given the latitude dependence of f their planetary vorticity therefore increases. Since absolute vorticity is conserved, however, relative vorticity must decrease locally. On the western side conditions are opposite: The southward flow

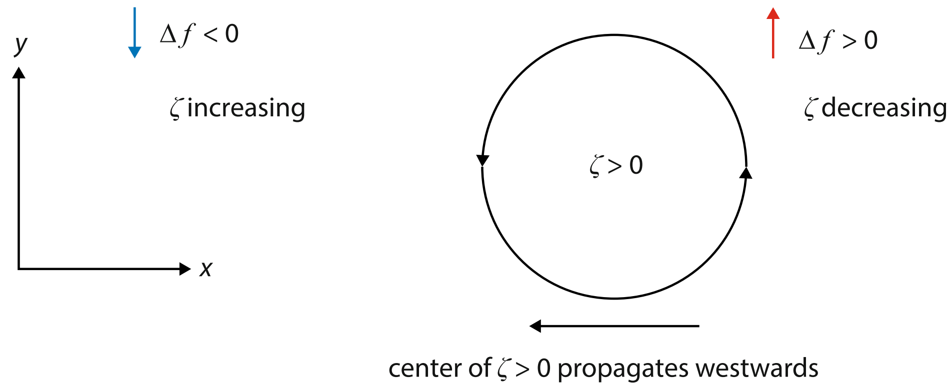

Fig. 5.9 For the illustration of the dynamics of short-wave Rossby waves, a top view of a positive vorticity anomaly (low-pressure system) in a wave on the northern hemisphere. The northwards-directed flow at the eastern side leads, via absolute-vorticity conservation, to a reduction of relative vorticity, while the southward flow on the western side raises the relative vorticity. The result is a westward propagation of the vortex

in combination with absolute-vorticity conservation leads to a local increase of relative vorticity. As a result the vorticity anomaly propagates westwards. Therefore the whole chain of high- and low-pressure systems propagates westwards, just as deduced above from the phase velocity.

Long Wavelengths ($K^2 \ll 1/L_d^2$): In this case the local change of relative vorticity (1) is negligible compared to the vortex-tube stretching. We thus have approximately

$$-\frac{1}{L_d^2}\frac{\partial \psi'}{\partial t} + \beta \frac{\partial \psi'}{\partial x} \approx 0 \tag{5.226}$$

which corresponds to the linear approximation of

$$\frac{D}{Dt}\left(\frac{f}{h}\right) = 0 \tag{5.227}$$

Therefore in such waves vortex-tube stretching leads to a conservation of the ratio between surface fluctuations and planetary vorticity. Consider for illustration a northern-hemisphere positive pressure anomaly, with positive η', as in Fig. 5.10. This is a positive streamfunction anomaly. On the eastern flank air masses move southwards, i.e., planetary vorticity decreases. As a consequence the streamfunction anomaly must decrease. On the western flank conditions are opposite: The northward-directed flow leads to an increase of the streamfunction anomaly. As a result the anomaly propagates westwards. In low-pressure systems

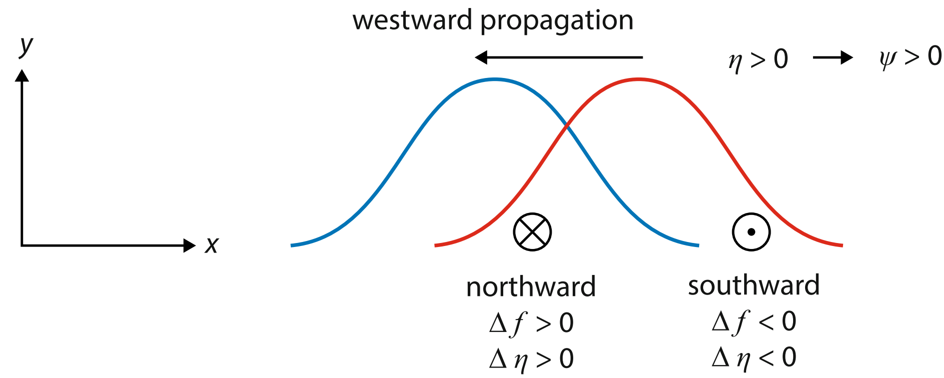

Fig. 5.10 For an illustration of the dynamics of long-wave Rossby waves, the side view of a positive streamfunction anomaly (high pressure system) in a wave on the northern hemisphere. The southward-directed flow on the eastern flank leads locally, due to vortex-tube stretching, to a decrease of the streamfunction anomaly, while the northward flow on the western flank increases the streamfunction anomaly. The result is a westward propagation of the vortex

the same net effect is obtained, so the chain of high- and low-pressure systems propagates westwards.

5.4.4 Summary

Already in their comparatively simple formulation the shallow-water equations admit essential wave solutions characterizing atmospheric variability on various time scales.

- In a *perturbation approach* one considers *infinitesimally small deviations* from an equilibrium atmosphere at rest.
- On the f*-plane* one obtains, via *Fourier transformation* in space and time, as only non-trivial solutions *steady geostrophic flow* and *external inertia-gravity waves* with the corresponding *dispersion relations* and *polarization relations*.
- This example has also been used for introducing the important concepts of *phase velocity* and *group velocity*.
- On the β*-plane* with latitude dependent Coriolis parameter the various meridional wave numbers are not decoupled any more, so that spatial Fourier transformation does not help. This can, however, be achieved within the framework of quasigeostrophic theory, holding for small Rossby numbers, i.e., large time scales in comparison with the inertial period. Via Fourier transformation one obtains *Rossby waves*.
- The *dynamics of Rossby waves* can be well understood on the basis of *potential-vorticity conservation*.

5.5 Geostrophic Adjustment

Geostrophic equilibrium is a ubiquitous phenomenon on the weather map. Beyond the mere diagnostics the question arises what happens if an atmosphere is on the synoptic scales not to leading order in geostrophic equilibrium. In fact this will lead to an adjustment process by which equilibration is achieved. An important factor in this is that gravity waves have a considerably larger group velocity than Rossby waves. An initial non-geostrophic structure, composed of Rossby waves and gravity waves, will radiate gravity waves which will leave the geostrophically balanced Rossby-wave part behind. This process, and the determination of the geostrophically balanced final state resulting from an arbitrary initial state, shall be our focus here. For simplicity we restrict ourselves to the case of linear dynamics on an f-plane.

5.5.1 The General Solution of the Linear Shallow-Water Equations on an f Plane

We consider the linearized equations (5.148) and (5.149) for infinitesimally small perturbations of a reference state at rest, while neglecting the β effect:

$$\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{f}_0 \times \mathbf{u}' = -g\nabla\eta' \tag{5.228}$$

$$\frac{\partial \eta'}{\partial t} + H\nabla\cdot\mathbf{u}' = 0 \tag{5.229}$$

The sum $H\mathbf{u}'\cdot$ (5.228) $+g\eta'$ (5.229) yields the conservation relation

$$\frac{\partial e_P}{\partial t} + \nabla\cdot\left(gH\eta'\mathbf{u}'\right) = 0 \tag{5.230}$$

with the *pseudo-energy density*

$$e_P = \left(H\frac{|\mathbf{u}'|^2}{2} + g\frac{\eta'^2}{2}\right) \tag{5.231}$$

Integration of (5.230) via the Gauss theorem shows that under typical boundary conditions, e.g., periodic boundary conditions or no normal flow, *pseudo-energy*

$$E_P = \int dx \int dy e_P \tag{5.232}$$

is conserved:

$$\frac{dE_P}{dt} = 0 \tag{5.233}$$

Thus motivated we now define a vector field

$$\boldsymbol{\Psi}(\mathbf{x},t) = \begin{pmatrix} \sqrt{H}u' \\ \sqrt{H}v' \\ \sqrt{g}\eta' \end{pmatrix} (\mathbf{x},t) \tag{5.234}$$

whose half norm coincides with its pseudo-energy, i.e.,

$$E_P = \frac{1}{2}\int dx \int dy\, |\boldsymbol{\Psi}|^2 \tag{5.235}$$

Multiplying the linearized momentum equation (5.228) by $\sqrt{H}$ and the linearized continuity equation (5.229) by $\sqrt{g}$ yields component-wise the system

$$\frac{\partial}{\partial t}\Psi_1 - f_0\Psi_2 = -c\frac{\partial \Psi_3}{\partial x} \tag{5.236}$$

$$\frac{\partial}{\partial t}\Psi_2 + f_0\Psi_1 = -c\frac{\partial \Psi_3}{\partial y} \tag{5.237}$$

$$\frac{\partial}{\partial t}\Psi_3 + c\left(\frac{\partial \Psi_1}{\partial x} + \frac{\partial \Psi_2}{\partial y}\right) = 0 \tag{5.238}$$

where

$$c = \sqrt{gH} \tag{5.239}$$

is the magnitude of the phase velocity of high-frequent gravity waves.

Now we write $\boldsymbol{\Psi}$ as Fourier integral in space

$$\boldsymbol{\Psi}(\mathbf{x},t) = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl\, \hat{\boldsymbol{\Psi}}(\mathbf{k},t)\, e^{i\mathbf{k}_h\cdot\mathbf{x}} \tag{5.240}$$

The spatial Fourier transform of (5.236)–(5.238) yields

$$\frac{\partial \hat{\Psi}_1}{\partial t} - f_0\hat{\Psi}_2 = -ikc\hat{\Psi}_3 \tag{5.241}$$

$$\frac{\partial \hat{\Psi}_2}{\partial t} + f_0\hat{\Psi}_1 = -ilc\hat{\Psi}_3 \tag{5.242}$$

$$\frac{\partial \hat{\Psi}_3}{\partial t} + ic\left(k\hat{\Psi}_1 + l\hat{\Psi}_2\right) = 0 \tag{5.243}$$

or

$$i\frac{\partial \hat{\boldsymbol{\Psi}}}{\partial t} = H\hat{\boldsymbol{\Psi}} \tag{5.244}$$

where

$$H = \begin{pmatrix} 0 & if_0 & kc \\ -if_0 & 0 & lc \\ kc & lc & 0 \end{pmatrix} \tag{5.245}$$

is the operator of the linear equation system. Its Hermitian nature $H^\dagger = H$ directly corresponds to pseudo-energy conservation. Here $H^\dagger$ indicates the complex-conjugate transpose of H.

Finally we also do a Fourier transform in time, i.e., we set

$$\hat{\boldsymbol{\Psi}}(\mathbf{k}, t) = \int_{-\infty}^{\infty} d\omega\, \tilde{\boldsymbol{\Psi}}(\mathbf{k}, \omega)\, e^{-i\omega t} \tag{5.246}$$

The temporal Fourier transform of (5.244) yields

$$\omega\tilde{\boldsymbol{\Psi}} = H\tilde{\boldsymbol{\Psi}} \tag{5.247}$$

Thus, the only non-vanishing solutions must be eigenvectors of H. As 3×3 matrix the latter has at given wave vector $\mathbf{k}$ three eigenvalues ω_α $(\alpha = 1, 2, 3)$ and corresponding eigenvectors $\mathbf{A}^\alpha$. Hence

$$\tilde{\boldsymbol{\Psi}}(\mathbf{k}, \omega) = \sum_{\alpha=1}^{3} \mathbf{A}^\alpha(\mathbf{k})\, \delta\left[\omega - \omega_\alpha(\mathbf{k})\right] \tag{5.248}$$

where for each α

$$H\mathbf{A}^\alpha = \omega_\alpha \mathbf{A}^\alpha \tag{5.249}$$

The eigenvalues or *eigenfrequencies* can be found as zeroes of the characteristic polynomial

$$0 = \det(H - \omega I) = \det\begin{pmatrix} -\omega & if_0 & kc \\ -if_0 & -\omega & lc \\ kc & lc & -\omega \end{pmatrix} = \omega\left[f_0^2 + c^2(k^2 + l^2) - \omega^2\right] \tag{5.250}$$

They are

$$\omega_1 = 0 \tag{5.251}$$

$$\omega_{2,3} = \pm\sqrt{f_0^2 + c^2 K^2} \tag{5.252}$$

Clearly the first eigenfrequency is the one of the geostrophic mode, while the two others correspond to gravity waves.

The structure of the solutions is obtained by inserting the respective eigenfrequency into (5.249) and solving for two of the fields, given the third. For example, inserting $\omega_1 = 0$ yields

$$if_0 A_2^1 + kcA_3^1 = 0 \tag{5.253}$$

$$-if_0 A_1^1 + lcA_3^1 = 0 \tag{5.254}$$

$$kcA_1^1 + lcA_2^1 = 0 \tag{5.255}$$

thus

$$A_1^1 = -i\frac{lc}{f_0}A_3^1 \tag{5.256}$$

$$A_2^1 = i\frac{kc}{f_0}A_3^1 \tag{5.257}$$

or

$$\mathbf{A}^1 = a_1 \begin{pmatrix} -ilc \\ ikc \\ f_0 \end{pmatrix} \tag{5.258}$$

where the normalization factor $a_1(\mathbf{k})$ is chosen such that

$$\left|\mathbf{A}^1\right|^2 = \left\langle \mathbf{A}^1, \mathbf{A}^1 \right\rangle = 1 \tag{5.259}$$

Here we define the euclidian scalar product between two vectors $\mathbf{X}$ and $\mathbf{Y}$ as

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \mathbf{X}^\dagger \mathbf{Y} = X_i^* Y_i \tag{5.260}$$

X_i^* is the complex conjugate of X_i. In analogy we find

$$\mathbf{A}^{2,3} = a_{2,3} \begin{bmatrix} \omega_{2,3}k + if_0 l \\ \omega_{2,3}l - if_0 k \\ c\left(k^2 + l^2\right) \end{bmatrix} \tag{5.261}$$

One can check easily that, at appropriate choice of the normalization factors, the eigenvectors are orthonormal, i.e.,

$$\left\langle \mathbf{A}^\alpha, \mathbf{A}^\beta \right\rangle = \delta_{\alpha\beta} \tag{5.262}$$

For $\alpha = \beta$ this is already guaranteed by the choice of the a_α. Furthermore, from the Hermitian nature of H follows

$$\omega_\beta \langle \mathbf{A}^\alpha, \mathbf{A}^\beta \rangle = \left(\mathbf{A}^\alpha\right)^\dagger H\mathbf{A}^\beta = \left(H\mathbf{A}^\alpha\right)^\dagger \mathbf{A}^\beta = \omega_\alpha \left(\mathbf{A}^\alpha\right)^\dagger \mathbf{A}^\beta = \omega_\alpha \langle \mathbf{A}^\alpha, \mathbf{A}^\beta \rangle \tag{5.263}$$

since all ω_α are real. If $\alpha \neq \beta$ then $\omega_\alpha \neq \omega_\beta$ and therefore also $\langle \mathbf{A}^\alpha, \mathbf{A}^\beta \rangle = 0$ which shows (5.262).

Since the eigenvectors $\mathbf{A}^\alpha$ form a complete basis, the time-dependent Fourier transform $\tilde{\boldsymbol{\Psi}}$ can at each instant be decomposed into contributions from these vectors, i.e.,

$$\tilde{\boldsymbol{\Psi}}(\mathbf{k}, t) = \sum_{\alpha=1}^{3} C_\alpha(\mathbf{k}, t)\, \mathbf{A}^\alpha(\mathbf{k}) \tag{5.264}$$

The corresponding coefficients are obtained by projecting $\tilde{\boldsymbol{\Psi}}$ onto the eigenvectors so that

$$C_\alpha = \left\langle \mathbf{A}^\alpha, \tilde{\boldsymbol{\Psi}} \right\rangle \tag{5.265}$$

Inserting (5.264) into (5.244) yields, also using the eigenvalue equation (5.249),

$$i \sum_{\alpha=1}^{3} \frac{\partial C_\alpha}{\partial t} \mathbf{A}^\alpha = \sum_{\alpha=1}^{3} C_\alpha H \mathbf{A}^\alpha = \sum_{\alpha=1}^{3} \omega_\alpha C_\alpha \mathbf{A}^\alpha \tag{5.266}$$

Thus we have

$$\frac{\partial C_\alpha}{\partial t} = -i\omega_\alpha C_\alpha \tag{5.267}$$

This is solved by

$$C_\alpha (\mathbf{k}, t) = D_\alpha (\mathbf{k})\, e^{-i\omega_\alpha(\mathbf{k})t} \tag{5.268}$$

where

$$D_\alpha (\mathbf{k}) = C_\alpha (\mathbf{k}, 0) \tag{5.269}$$

is determined from the initial condition, i.e.,

$$\boxed{D_\alpha (\mathbf{k}) = \left\langle \mathbf{A}^\alpha (\mathbf{k}) , \tilde{\mathbf{\Psi}} (\mathbf{k}, 0) \right\rangle} \tag{5.270}$$

The general solution of the linear equations therefore is

$$\boxed{\tilde{\mathbf{\Psi}} (\mathbf{k}, t) = \sum_{\alpha=1}^{3} D_\alpha (\mathbf{k})\, \mathbf{A}^\alpha (\mathbf{k})\, e^{-i\omega_\alpha(\mathbf{k})t}} \tag{5.271}$$

or

$$\boxed{\mathbf{\Psi} (\mathbf{x}, t) = \sum_{\alpha=1}^{3} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl\, D_\alpha (\mathbf{k})\, \mathbf{A}^\alpha (\mathbf{k})\, e^{i[\mathbf{k}\cdot\mathbf{x} - \omega_\alpha(\mathbf{k})t]}} \tag{5.272}$$

It thus always is a superposition of the geostrophic mode and the gravity waves. The latter are not geostrophically balanced.

5.5.2 The Adjustment Process

The Process in General

We consider an initial state which differs from a trivial equilibrium at rest, with constant η', only in a region of finite extent. Most conceivable states of this kind are not in geostrophic equilibrium and thus also carry inertia-gravity wave contributions. Those, however, have a non-vanishing group velocity $\mathbf{c}_{2,3,g} \neq 0$, while the geostrophic part does not move since $\mathbf{c}_{1,g} = 0$. The gravity waves therefore will leave the region under consideration toward infinity. Only the geostrophic part remains so that the final state is, at $t \to \infty$,

$$\tilde{\mathbf{\Psi}} (\mathbf{k}, t \to \infty) = C_1 (\mathbf{k}, t \to \infty)\, \mathbf{A}^1 (\mathbf{k}) \tag{5.273}$$

and thus in geostrophic equilibrium. Due to the orthogonality of the $\mathbf{A}^\alpha$ we get for $\alpha = 2, 3$

$$\left\langle \mathbf{A}^\alpha(\mathbf{k}), \tilde{\mathbf{\Psi}}(\mathbf{k}, t \to \infty) \right\rangle = 0 \tag{5.274}$$

Evaluation of the two scalar products yields

$$(\omega_2 k - i f_0 l)\, \tilde{\Psi}_1 + (\omega_2 l + i f_0 k)\, \tilde{\Psi}_2 + c\left(k^2 + l^2\right) \tilde{\Psi}_3 = 0 \tag{5.275}$$
$$(\omega_3 k - i f_0 l)\, \tilde{\Psi}_1 + (\omega_3 l + i f_0 k)\, \tilde{\Psi}_2 + c\left(k^2 + l^2\right) \tilde{\Psi}_3 = 0 \tag{5.276}$$

Eliminating $\hat{\Psi}_1$ from these two equations, by forming the difference $(\omega_3 k - i f_0 l)$ (5.275) $-\, (\omega_2 k - i f_0 l)$ (5.276), gives

$$i f_0 \left(k^2 + l^2\right)(\omega_3 - \omega_2)\, \tilde{\Psi}_2 + ck\left(k^2 + l^2\right)(\omega_3 - \omega_2)\, \tilde{\Psi}_3 = 0 \tag{5.277}$$

or

$$i f_0 \tilde{\Psi}_2 = -ck\tilde{\Psi}_3 \tag{5.278}$$

From the Fourier transform of $\mathbf{\Psi}$ we obtain those of $\mathbf{u}$ and η via (5.234) and then use the definition of c according to (5.239), finally obtaining

$$f_0 \tilde{v} = igk\tilde{\eta} \tag{5.279}$$

The inverse Fourier transform of this equation is

$$f_0 v' = g\frac{\partial \eta'}{\partial x} \tag{5.280}$$

This just verifies that the meridional wind is in geostrophic balance. Likewise, eliminating $\hat{\Psi}_2$ from (5.275) and (5.276) yields

$$-\, f_0 u' = g\frac{\partial \eta'}{\partial y} \tag{5.281}$$

which verifies that also the zonal wind is in geostrophic balance. Certainly these results could also have been determined directly from the structure of the geostrophic eigenvector in (5.258).

For the determination of the structure of the final state we need $C_1(\mathbf{k}, t \to \infty)$ in (5.273). However, since $\omega_1 = 0$, we have with (5.268)

$$C_1(\mathbf{k}, t \to \infty) = C_1(\mathbf{k}, 0) \tag{5.282}$$

or

$$\left\langle \mathbf{A}^1(\mathbf{k}), \tilde{\mathbf{\Psi}}(\mathbf{k}, t \to \infty) \right\rangle = \left\langle \mathbf{A}^1(\mathbf{k}), \tilde{\mathbf{\Psi}}(\mathbf{k}, 0) \right\rangle \tag{5.283}$$

The corresponding conservation quantity is

$$\left\langle \mathbf{A}^1(\mathbf{k}), \tilde{\boldsymbol{\Psi}}(\mathbf{k}, t) \right\rangle = a_1^*(\mathbf{k}) \left(ilc\tilde{\Psi}_1 - ikc\tilde{\Psi}_2 + f_0\tilde{\Psi}_3 \right) = a_1^*(\mathbf{k}) \sqrt{g} H \left(il\tilde{u} - ik\tilde{v} + \frac{f_0}{H}\tilde{\eta} \right) \tag{5.284}$$

where in a first step we have used the definition of $\mathbf{A}^1$ according to (5.258), and in a second we obtained from the Fourier transform of Ψ_i via (5.234) those of $\mathbf{u}$ and η, also using the definition (5.239) of c. Thus $-\left[il\tilde{u} - ik\tilde{v} + (f_0/H)\tilde{\eta}\right]$ is conserved. The inverse Fourier transform of this, however, is the quasigeostrophic potential vorticity

$$\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} - \frac{f_0}{H}\eta' \right) = \left(\zeta' - \frac{f_0}{H}\eta' \right) \tag{5.285}$$

The geostrophic final state can therefore be determined from the conservation of quasi-geostrophic potential vorticity:

$$\boxed{\left(\zeta' - \frac{f_0}{H}\eta' \right)(t \to \infty) = \left(\zeta' - \frac{f_0}{H}\eta' \right)(t = 0)} \tag{5.286}$$

Note that the final state satisfies, due to its geostrophy,

$$u' = -\frac{g}{f_0}\frac{\partial \eta'}{\partial y} \tag{5.287}$$

$$v' = \frac{g}{f_0}\frac{\partial \eta'}{\partial x} \tag{5.288}$$

and thus

$$\zeta' = \frac{g}{f_0}\nabla^2\eta' \tag{5.289}$$

Therefore the surface fluctuations can be determined for $t \to \infty$ from

$$\boxed{\left(\nabla^2\eta' - \frac{\eta'}{L_d^2} \right)(t \to \infty) = \left(\frac{f_0}{g}\zeta' - \frac{\eta'}{L_d^2} \right)(t = 0)} \tag{5.290}$$

Here the Rossby deformation radius is defined by (5.85). After having solved (5.290) with appropriate boundary conditions the corresponding wind can be determined from (5.287) and (5.288).

Geostrophic Adjustment of a Pressure Jump

For further illustration we here consider a classic example. The initial state is at rest but at $x = 0$ it has, as shown in Fig. 5.11, a discontinuity in the surface fluctuations or the pressure:

$$\mathbf{u}'(\mathbf{x}, 0) = 0 \tag{5.291}$$

$$\eta'(\mathbf{x}, 0) = -\eta_0 \,\mathrm{sgn}\,(x) \tag{5.292}$$

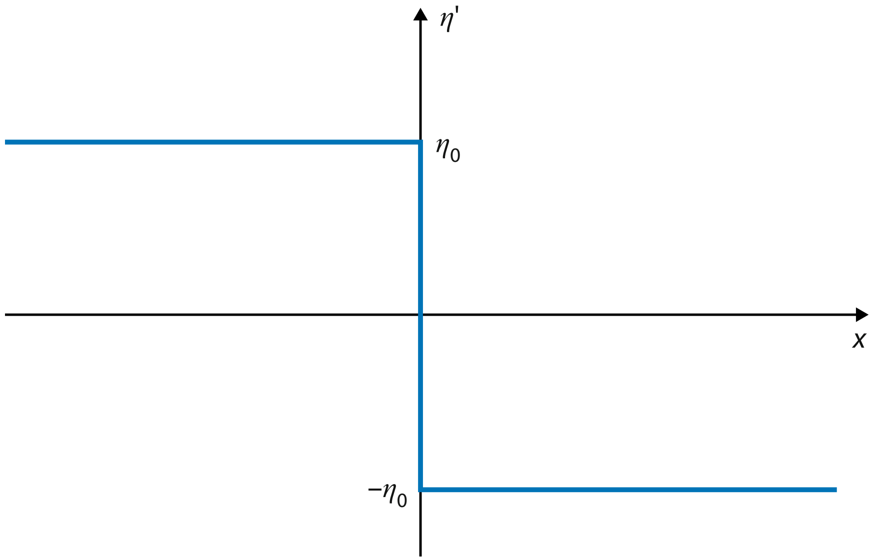

Fig. 5.11 Pressure jump, only depending on the zonal direction x, which is not in geostrophic equilibrium with an atmosphere at rest

Because the initial state only depends on x, also the one developing from it cannot depend on y. Moreover, the initial state is at rest, so that we have on the right-hand side of (5.290) for the relative vorticity $\zeta' = 0$. It therefore is

$$\frac{d^2\eta'}{dx^2} - \frac{\eta'}{L_d^2} = \frac{\eta_0}{L_d^2}\,\mathrm{sgn}(x) \tag{5.293}$$

This ordinary differential equation can be solved by standard means. One first must determine a special solution η'_s of the equation, and the general solution η'_h of the corresponding homogeneous equation

$$\frac{d^2\eta'_h}{dx^2} - \frac{\eta'_h}{L_d^2} = 0 \tag{5.294}$$

The final solution is a sum of η'_s and η'_h, where the free coefficients in the second part are determined from the boundary conditions. In the present case the latter demand that η' must not diverge as $x \to \pm\infty$. Beyond this we also demand that η' and $d\eta'/dx$ be continuous at $x = 0$ so that $d^2\eta'/dx^2$ there is well-defined. These are also the minimal conditions for the existence of ζ' at $x = 0$.

As special solution of the general equation we try the ansatz that η'_s be constant everywhere besides at $x = 0$. This means $d^2\eta'/dx^2 = 0$. We obtain

$$\eta'_s = -\eta_0\,\mathrm{sgn}(x) \tag{5.295}$$

Indeed this is a solution of (5.293).

The homogeneous equation has only constant coefficients. One therefore can use the ansatz $\eta_h' = \exp(mx)$, yielding $m = \pm 1/L_d$. The general solution of the homogeneous problem thus is

$$\eta_h' = a_+ \, e^{x/L_d} + a_- \, e^{-x/L_d} \tag{5.296}$$

For positive and negative x we now first must determine the respective coefficients $a_\pm$ in the complete solution

$$\eta' = \eta_s' + \eta_h' \tag{5.297}$$

so that it does not diverge as $x \to \pm\infty$. One obtains

$$\eta' = -\eta_0 \, \mathrm{sgn}(x) + \begin{cases} a_+ \, e^{x/L_d} & \text{at } x < 0 \\ a_- \, e^{-x/L_d} & \text{at } x > 0 \end{cases} \tag{5.298}$$

Finally we turn to the requirement that η' and $d\eta'/dx$ be continuous at $x = 0$. From (5.298) follows

$$\frac{d\eta'}{dx} = \begin{cases} \dfrac{a_+}{L_d} \, e^{x/L_d} & \text{at } x < 0 \\ -\dfrac{a_-}{L_d} \, e^{-x/L_d} & \text{at } x > 0 \end{cases} \tag{5.299}$$

Thus

$$\lim_{x \to 0} \eta' = \begin{cases} \eta_0 + a_+ & \text{at } x < 0 \\ -\eta_0 + a_- & \text{at } x > 0 \end{cases} \tag{5.300}$$

$$\lim_{x \to 0} \frac{d\eta'}{dx} = \begin{cases} \dfrac{a_+}{L_d} & \text{at } x < 0 \\ -\dfrac{a_-}{L_d} & \text{at } x > 0 \end{cases} \tag{5.301}$$

Therefore continuity of $d\eta'/dx$ implies

$$a_+ = -a_- \tag{5.302}$$

Continuity of η' leads to

$$\eta_0 + a_+ = -\eta_0 + a_- \tag{5.303}$$

Together with (5.302) this yields

$$a_+ = -\eta_0 \tag{5.304}$$

$$a_- = \eta_0 \tag{5.305}$$

so that finally

$$\eta'(\mathbf{x}, t \to \infty) = \eta_0 \begin{cases} \left(1 - e^{x/L_d}\right) & \text{at } x < 0 \\ \left(e^{-x/L_d} - 1\right) & \text{at } x > 0 \end{cases} \tag{5.306}$$

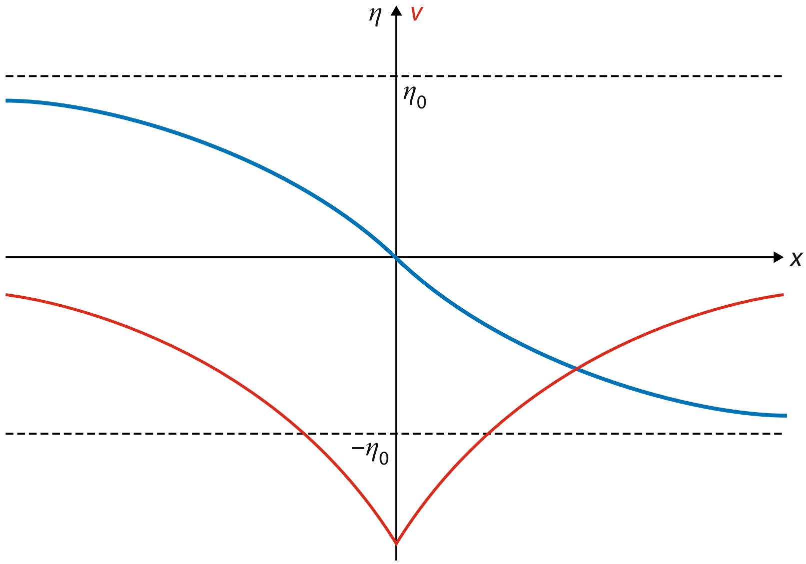

Fig. 5.12 Distribution of η' and v', resulting after the geostrophic adjustment of the pressure jump from Fig. 5.11

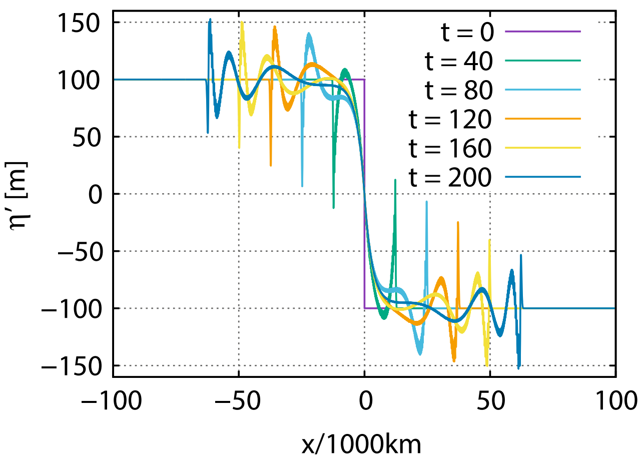

Fig. 5.13 Temporal development (time in units of 1000 s) of the geostrophic adjustment of a pressure jump. Note the gravity-wave fronts moving outwards while a geostrophically balanced state remains in the center

The geostrophic wind resulting from this with (5.287) and (5.288) is purely meridional, i.e., $u' = 0$ and

$$v'(\mathbf{x}, t \to \infty) = -\frac{\eta_0 g}{f_0 L_d} \begin{cases} e^{x/L_d} & \text{at } x < 0 \\ e^{-x/L_d} & \text{at } x > 0 \end{cases} \tag{5.307}$$

One obtains a meridional jet with maximal intensity at $x = 0$. The distribution of η' and v' is shown in Fig. 5.12, while Fig. 5.13 illustrates the temporal development of the adjustment process. For the qualitative illustration of the radiation of gravity waves from strong pressure gradients we show in Fig. 5.14 a snapshot of geopotential and horizontal divergence on the 200mbar surface over Europe.

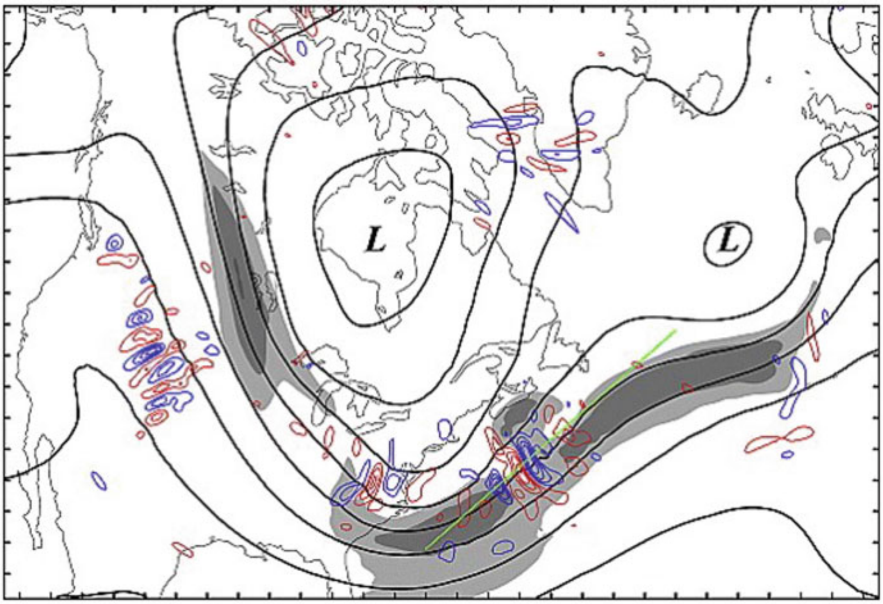

Fig. 5.14 For the qualitative demonstration of the relevance of the geostrophic adjustment process, a snapshot of horizontal divergence (red and blue) on the 80mbar pressure level and geopotential (black) at 300mb over North America and the North Atlantic. Shading indicates the wind strength. In pressure coordinates the geopotential plays the same role as elsewhere pressure as streamfunction in geostrophic scaling. Since the geostrophic flow has zero horizontal divergence the latter can be used very well as indicator of gravity-wave activity. Note the increased gravity-wave intensity in the vicinity of strong gradients of the geopotential, where the dynamics tends to deviate from geostrophic scaling. Figure from Wu and Zhang (2004)

5.5.3 Summary

An initially non-geostrophic state has contributions both from Rossby waves and from gravity waves. Since the latter have a much larger group velocity they are radiated away so that finally only the geostrophically balanced Rossby-wave part remains.

- For an analysis of this process we consider the limit of *linear dynamics on the f-plane*.
- In a general solution of the *initial-value problem* one can show that at any time the state is constituted by a steady geostrophic part and propagating inertia-gravity waves. The corresponding contributions are obtained via projection of the initial state onto these eigenmodes.
- The finally remaining part can be obtained from *potential-vorticity conservation*. The potential vorticity of the final state yields the streamfunction or surface elevation by quasigeostrophic theory.

5.6 Recommendations for Further Reading

Excellent texts on shallow-water dynamics are the books by Pedlosky (1987), Salmon (1998), Vallis (2006), and Zeitlin (2018).