6 Quasigeostrophic Dynamics of the Stratified Atmosphere

Strictly spoken the shallow-water equations only hold under conditions not valid for the atmosphere. Both the assumption of a constant density and the one of vanishing vertical gradients in the horizontal wind are not realistic. These will now be dropped. We will not, however, treat the atmosphere in full generality but rather focus on the *synoptic scales* and derive the corresponding quasigeostrophic theory. This will enable us to describe not only the *vertical structure and vertical propagation of Rossby waves* but also the generation of synoptic-scale extratropical weather by *baroclinic instability*.

6.1 Quasigeostrophic Theory and Its Potential Vorticity

6.1.1 Analysis of Momentum and Continuity Equation

Scale Analysis

Much in the derivation of the quasigeostrophic theory of the baroclinic atmosphere resembles the corresponding theory for the shallow-water equations. In addition to there we here split the thermodynamic fields into a part from a hydrostatic reference atmosphere at rest, with only vertical spatial dependence, and the deviations therefrom. We thus write, with $z = r - a$,

$$\rho = \overline{\rho}(z) + \tilde{\rho}(\lambda, \phi, z, t) \tag{6.1}$$

and

$$p = \overline{p}(z) + \tilde{p}(\lambda, \phi, z, t) \tag{6.2}$$

where the reference-atmosphere part satisfies

$$\frac{d\overline{p}}{dz} = -g\overline{\rho} \tag{6.3}$$

U. Achatz, *Atmospheric Dynamics*, https://doi.org/10.1007/978-3-662-63941-2_6

We also demand that, in agreement with observations, the density deviations from the density of the reference atmosphere are small:

$$|\tilde{\rho}| \ll \overline{\rho} \tag{6.4}$$

For reasons which will become clear below we choose an altitude-dependent scaling for the pressure and density fluctuations:

$$\tilde{p} = \mathcal{P}(z)\,\hat{p} \tag{6.5}$$

$$\tilde{\rho} = \mathcal{R}(z)\,\hat{\rho} \tag{6.6}$$

Following (6.4) we have $\mathcal{R} \ll \overline{\rho}$.

As already in quasigeostrophic shallow-water theory we introduce a horizontal length scale $L = 10^3$ km and a horizontal-wind scale $U = 10$ m/s so that, with some reference longitude and latitude λ_0 and ϕ_0, respectively, and consistent with the estimates (5.38) and (5.39)

$$\begin{pmatrix} \lambda \\ \phi \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \phi_0 \end{pmatrix} + \frac{L}{a}\begin{pmatrix} \hat{\lambda} \\ \hat{\phi} \end{pmatrix} \tag{6.7}$$

and

$$\mathbf{u} = U\hat{\mathbf{u}} \tag{6.8}$$

The time scale is again

$$t = \frac{L}{U}\hat{t} \tag{6.9}$$

In addition we introduce a vertical length scale $H = 10$ km and a scale W for the vertical wind so that

$$z = H\hat{z} \tag{6.10}$$

$$w = W\hat{w} \tag{6.11}$$

The vertical length scale approximately corresponds to the height of typical synoptic-scale weather structures, but also to the vertical extent of the troposphere and its hydrostatic scale height. The vertical-wind scale can be related to the horizontal-wind scale via the continuity equation

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0 \tag{6.12}$$

Due to (6.4) the latter is approximately

$$w\frac{d\overline{\rho}}{dz} + \overline{\rho}\nabla\cdot\mathbf{v} = 0 \tag{6.13}$$

or, in local Cartesian coordinates,

$$\nabla \cdot \mathbf{u} + \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} (\overline{\rho} w) = 0 \tag{6.14}$$

The non-dimensionalization of this equation yields

$$\frac{U}{L} \hat{\nabla} \cdot \hat{\mathbf{u}} + \frac{W}{H} \frac{1}{\overline{\rho}} \frac{\partial}{\partial \hat{z}} \left(\overline{\rho} \hat{w}\right) = 0 \tag{6.15}$$

For an equilibration between the two terms one needs

$$W = \frac{H}{L} U \tag{6.16}$$

Therefore the vertical winds must be weaker than the horizontal winds by at least two orders of magnitude. Below we will see that this is actually only an estimate of an upper bound.

No we turn to the two horizontal-momentum equations. Because of $r = a + z$ and (6.10) one has

$$r = a \left(1 + \frac{H}{a} \hat{z}\right) \tag{6.17}$$

where $H/a \approx 10^{-3} \ll 1$. Inserting this together with (6.7–6.11) and (6.16) into the material derivative (1.107) yields

$$\frac{D}{Dt} = \frac{U}{L} \frac{\tilde{D}}{D\hat{t}} \tag{6.18}$$

with the non-dimensional material derivative

$$\frac{\tilde{D}}{D\hat{t}} = \frac{\partial}{\partial \hat{t}} + \frac{\hat{u}}{\left(1 + \frac{H}{a} \hat{z}\right) \cos \phi} \frac{\partial}{\partial \hat{\lambda}} + \frac{\hat{v}}{1 + \frac{H}{a} \hat{z}} \frac{\partial}{\partial \hat{\phi}} + \hat{w} \frac{\partial}{\partial \hat{z}} \tag{6.19}$$

Moreover, one has

$$2\Omega \sin \phi = f = f_0 \hat{f} \tag{6.20}$$

with

$$f_0 = 2\Omega \sin \phi_0 \tag{6.21}$$

$$\hat{f} = \frac{\sin \phi}{\sin \phi_0} \tag{6.22}$$

and

$$2\Omega \cos \phi = a\beta \frac{\cos \phi}{\cos \phi_0} \tag{6.23}$$

with

$$\beta = \frac{2\Omega \cos \phi_0}{a} = \frac{f_0}{L} Ro \hat{\beta} \tag{6.24}$$

$$\hat{\beta} = \frac{1}{Ro} \frac{L}{a} \cot \phi_0 = \mathcal{O}(1) \tag{6.25}$$

Non-dimensionalization of the horizontal-momentum equations in (1.105) via (6.1), (6.2), (6.5–6.11), and (6.16) yields, also using (6.17) and (6.18),

$$Ro\frac{\tilde{D}\hat{u}}{D\hat{t}} - \frac{L}{a}Ro\frac{\hat{u}\hat{v}}{1+\frac{H}{a}\hat{z}}\tan\phi + \frac{H}{L}\frac{L}{a}Ro\frac{\hat{u}\hat{w}}{1+\frac{H}{a}\hat{z}} - \hat{f}\hat{v} + \frac{H}{L}\frac{a}{L}Ro\hat{\beta}\frac{\cos\phi}{\cos\phi_0}\hat{w}$$
$$= -\frac{\mathcal{P}}{\overline{\rho}Lf_0U}\frac{1}{\left(1+\frac{\mathcal{R}}{\overline{\rho}}\hat{\rho}\right)}\frac{1}{\left(1+\frac{H}{a}\hat{z}\right)}\frac{1}{\cos\phi}\frac{\partial\hat{p}}{\partial\hat{\lambda}} \tag{6.26}$$

$$Ro\frac{\tilde{D}\hat{v}}{D\hat{t}} + \frac{L}{a}Ro\frac{\hat{u}^2}{1+\frac{H}{a}\hat{z}}\tan\phi + \frac{H}{L}\frac{L}{a}Ro\frac{\hat{u}\hat{w}}{1+\frac{H}{a}\hat{z}} + \hat{f}\hat{u}$$
$$= -\frac{\mathcal{P}}{\overline{\rho}Lf_0U}\frac{1}{\left(1+\frac{\mathcal{R}}{\overline{\rho}}\hat{\rho}\right)}\frac{1}{\left(1+\frac{H}{a}\hat{z}\right)}\frac{\partial\hat{p}}{\partial\hat{\phi}} \tag{6.27}$$

Since the Rossby number is with the chosen scaling $Ro = \mathcal{O}(10^{-1})$ the Coriolis term is the only term on the left-hand side of these equations which is not small. It can only be balanced by the pressure-gradient term on the right-hand side if

$$\boxed{\mathcal{P} = \overline{\rho}Lf_0U} \tag{6.28}$$

Thus we indeed obtain an altitude-dependent pressure scale so that

$$p = \overline{p}(z) + \overline{\rho}Lf_0U\hat{p} \tag{6.29}$$

In the treatment of the vertical-momentum equation in (1.105) we first rewrite the right-hand side via (6.1) and (6.2):

$$\frac{Dw}{Dt} - \frac{u^2+v^2}{r} - 2\Omega\cos\phi u = -\left[\frac{1}{\overline{\rho}+\tilde{\rho}}\left(\frac{d\overline{p}}{dz}+\frac{\partial\tilde{p}}{\partial z}\right)+g\right] \tag{6.30}$$

We further use the hydrostatic equilibrium (6.3) of the reference atmosphere and obtain

$$\frac{Dw}{Dt} - \frac{u^2+v^2}{r} - 2\Omega\cos\phi u = -\left[\frac{g\tilde{\rho}}{\overline{\rho}+\tilde{\rho}}+\frac{1}{\overline{\rho}+\tilde{\rho}}\frac{\partial\tilde{p}}{\partial z}\right] \tag{6.31}$$

Now we proceed as in the non-dimensionalization of the horizontal-momentum equations, also using (6.6), and obtain

$$
\frac{H}{L} Ro \frac{\tilde{D}\hat{w}}{D\hat{t}} - \frac{L}{a} Ro \frac{\hat{u}^2 + \hat{v}^2}{1 + \frac{H}{a}\hat{z}} - \frac{a}{L} Ro\hat{\beta} \frac{\cos\phi}{\cos\phi_0} \hat{u}
$$

$$
= -\frac{L}{H} \left[\frac{\hat{\rho}}{1 + \frac{\mathcal{R}}{\overline{\rho}}\hat{\rho}} \frac{\mathcal{R} g H}{\overline{\rho} f_0 U L} + \frac{1}{1 + \frac{\mathcal{R}}{\overline{\rho}}\hat{\rho}} \frac{1}{\overline{\rho}} \frac{\partial}{\partial \hat{z}} (\overline{\rho}\hat{p}) \right] \tag{6.32}
$$

Among the terms on the left-hand side the last is the largest. It is of order $\mathcal{O}(1)$ and thus still small compared to the factor $H/L \gg 1$ on the right-hand side. We conclude that at least to leading order the terms in the bracket on the right-hand side must cancel each other, which again only is possible if

$$
\boxed{\mathcal{R} = \frac{L f_0 U \overline{\rho}}{g H} = \overline{\rho} Ro \frac{L^2}{L_d^2}} \tag{6.33}
$$

One thus also sees that $\mathcal{R} \ll \overline{\rho}$, consistent with the basic assumptions. Moreover one obtains

$$
\rho = \overline{\rho} \left(1 + \overline{\rho} Ro \frac{L^2}{L_d^2} \hat{\rho} \right) \tag{6.34}
$$

With this choice the vertical-momentum equation becomes

$$
\frac{H}{L} Ro \frac{\tilde{D}\hat{w}}{D\hat{t}} - \frac{L}{a} Ro \frac{\hat{u}^2 + \hat{v}^2}{1 + \frac{H}{a}\hat{z}} - \frac{a}{L} Ro\hat{\beta} \frac{\cos\phi}{\cos\phi_0} \hat{u}
$$

$$
= -\frac{L}{H} \left[\frac{\hat{\rho}}{1 + Ro \frac{L^2}{L_d^2}\hat{\rho}} + \frac{1}{1 + Ro \frac{L^2}{L_d^2}\hat{\rho}} \frac{1}{\overline{\rho}} \frac{\partial}{\partial \hat{z}} (\overline{\rho}\hat{p}) \right] \tag{6.35}
$$

The non-dimensionalization of the continuity equation works the same way. We first non-dimensionalize the divergence in (1.109) as

$$
\nabla \cdot \mathbf{v} = \frac{U}{L} \left\{ \hat{\nabla} \cdot \hat{\mathbf{u}} + \frac{1}{\left(1 + \frac{H}{a}\hat{z}\right)^2} \frac{\partial}{\partial \hat{z}} \left[\left(1 + \frac{H}{a}\hat{z}\right)^2 \hat{w} \right] \right\}
$$

$$
= \frac{U}{L} \left[\hat{\nabla} \cdot \hat{\mathbf{u}} + \frac{\partial \hat{w}}{\partial \hat{z}} + \mathcal{O}\left(\frac{H}{a}\right) \right] \tag{6.36}
$$

with the non-dimensional horizontal divergence

$$\hat{\nabla} \cdot \hat{\mathbf{u}} = \frac{1}{1 + \frac{H}{a}\hat{z}} \left[\frac{1}{\cos\phi} \frac{\partial \hat{u}}{\partial \hat{\lambda}} + \frac{1}{\cos\phi} \frac{\partial}{\partial \hat{\phi}} \left(\cos\phi \hat{v}\right) \right] \tag{6.37}$$

This and (6.34) lead, in a non-dimensionalization closely analogous to what has been demonstrated above, to

$$Ro \frac{L^2}{L_d^2} \frac{\tilde{D}\hat{\rho}}{D\hat{t}} + \left(1 + Ro \frac{L^2}{L_d^2} \hat{\rho}\right) \left[\hat{\nabla} \cdot \hat{\mathbf{u}} + \frac{1}{\overline{\rho}} \frac{\partial}{\partial \hat{z}} (\overline{\rho} \hat{w}) + \mathcal{O}\left(\frac{H}{a}\right) \right] = 0 \tag{6.38}$$

Local Geometry and Characterization in Terms of Powers of the Rossby Number

From the definition (5.85) of the external Rossby deformation radius follows, with $H = 10\,\text{km}$ and $f_0 = 10^{-4}\,\text{s}^{-1}$, that

$$L_d \approx 3000\,\text{km} \tag{6.39}$$

But $L = 1000\,\text{km}$, so that we can use the central assumption

$$\boxed{\frac{L^2}{L_d^2} = \mathcal{O}(Ro)} \tag{6.40}$$

where at the given scaling $Ro = \mathcal{O}(10^{-1})$. One should note that this differs from the assumption $L^2/L_d^2 = \mathcal{O}(1)$ used in the derivation of shallow-water quasigeostrophic theory. With our choice of length scales one obtains moreover

$$\boxed{\frac{H}{L} = \mathcal{O}(Ro^2) \qquad \frac{L}{a} = \mathcal{O}(Ro) \qquad \frac{H}{a} = \mathcal{O}(Ro^3)} \tag{6.41}$$

Since $L/a = \mathcal{O}(Ro) \ll 1$, we expand the various trigonometric functions of the geographic latitude about the reference latitude and thus arrive at the local geometry of the β-plane. Thus, using (6.7),

$$\begin{aligned} \frac{1}{\cos\phi} \frac{\partial}{\partial \hat{\lambda}} &= \left\{ 1 + \tan\phi_0 \frac{L}{a} \hat{\phi} + \mathcal{O}\left[\left(\frac{L}{a} \right)^2 \right] \right\} \frac{1}{\cos\phi_0} \frac{\partial}{\partial \hat{\lambda}} \\ &= \left(1 + \frac{L}{a} \tan\phi_0\, \hat{y} \right) \frac{\partial}{\partial \hat{x}} + \mathcal{O}\left(Ro^2\right) \end{aligned} \tag{6.42}$$

where

$$(\hat{x}, \hat{y}) = \left(\cos\phi_0\, \hat{\lambda}, \hat{\phi} \right) \tag{6.43}$$

are the non-dimensional horizontal coordinates of the β-plane tangential at (λ_0, ϕ_0), as can be read via (5.40), (5.41), and $(x, y) = L(\hat{x}, \hat{y})$ from (6.7). Correspondingly one also has

$$\frac{\partial}{\partial\hat{\phi}} = \frac{\partial}{\partial\hat{y}} \tag{6.44}$$

so that the material derivative can be rewritten, also using

$$\frac{1}{1+\frac{H}{a}\hat{z}} = 1+\mathcal{O}\left(\frac{H}{a}\right) = 1+\mathcal{O}\left(Ro^3\right) \tag{6.45}$$

as

$$\frac{\tilde{D}}{D\hat{t}} = \frac{\hat{D}}{D\hat{t}} + Ro\frac{L/a}{Ro}\tan\phi_0\frac{\partial}{\partial\hat{x}} + \mathcal{O}\left(Ro^2\right) \tag{6.46}$$

with the definition

$$\frac{\hat{D}}{D\hat{t}} = \frac{\partial}{\partial\hat{t}} + \hat{u}\frac{\partial}{\partial\hat{x}} + \hat{v}\frac{\partial}{\partial\hat{y}} + \hat{w}\frac{\partial}{\partial\hat{z}} \tag{6.47}$$

Beyond this we have

$$\tan\phi = \mathcal{O}\left(1\right) \tag{6.48}$$

$$\frac{\cos\phi}{\cos\phi_0} = \mathcal{O}\left(1\right) \tag{6.49}$$

and, with (6.22) and $\hat{f}_0 = 1$,

$$\begin{aligned}\hat{f} &= \hat{f}_0 + \cot\phi_0\frac{L}{a}\hat{\phi} + \mathcal{O}\left[\left(\frac{L}{a}\right)^2\right]\\ &= \hat{f}_0 + Ro\hat{\beta}\,\hat{y} + \mathcal{O}\left(Ro^2\right)\end{aligned} \tag{6.50}$$

and

$$\frac{1}{1+\frac{\mathcal{R}}{\overline{\rho}}\hat{\rho}} = \frac{1}{1+Ro\frac{L^2}{L_d^2}\hat{\rho}} = 1+\mathcal{O}\left(Ro\frac{L^2}{L_d^2}\right) = 1+\mathcal{O}\left(Ro^2\right) \tag{6.51}$$

Using all these estimates we now rewrite the horizontal-momentum equations (6.26) and (6.27) so that only the larger terms are expressed explicitly that will be needed in the further treatment below. One obtains

$$\begin{aligned}&Ro\left[\frac{\hat{D}\hat{u}}{D\hat{t}} + \mathcal{O}\left(Ro\right)\right] - \left[\hat{f}_0 + Ro\hat{\beta}\hat{y} + \mathcal{O}\left(Ro^2\right)\right]\hat{v} + \mathcal{O}\left(Ro^2\right)\\ &\quad = -\left(1+\frac{L}{a}\tan\phi_0\,\hat{y}\right)\frac{\partial\hat{p}}{\partial\hat{x}} + \mathcal{O}\left(Ro^2\right)\end{aligned} \tag{6.52}$$

$$
\begin{aligned}
&Ro\left[\frac{\hat{D}\hat{v}}{D\hat{t}}+\mathcal{O}(Ro)\right]+\left[\hat{f}_0+Ro\hat{\beta}\hat{y}+\mathcal{O}\left(Ro^2\right)\right]\hat{u}+\mathcal{O}\left(Ro^2\right)\\
&=-\frac{\partial\hat{p}}{\partial\hat{y}}+\mathcal{O}\left(Ro^2\right)
\end{aligned}
\tag{6.53}
$$

Likewise the vertical-momentum equation (6.35) becomes

$$
\mathcal{O}(1)=-\frac{L}{H}\left[\hat{\rho}+\frac{1}{\overline{\rho}}\frac{\partial}{\partial\hat{z}}\left(\overline{\rho}\hat{p}\right)+\mathcal{O}\left(Ro^2\right)\right]\qquad\frac{L}{H}=\mathcal{O}\left(Ro^{-2}\right)
\tag{6.54}
$$

For the continuity equation (6.38) we first reformulate $\tilde{\nabla}\cdot\hat{\mathbf{u}}$ in (6.37). Expansion about the reference latitude ϕ_0 yields, with $\hat{\phi}=\hat{y}$,

$$
\frac{1}{\cos\phi}\frac{\partial}{\partial\hat{\phi}}\left(\cos\phi\,\hat{v}\right)=\frac{\partial\hat{v}}{\partial\hat{y}}-\frac{L}{a}\tan\phi_0\,\hat{v}+\mathcal{O}\left(Ro^2\right)
\tag{6.55}
$$

This together with (6.42) leads to

$$
\tilde{\nabla}\cdot\hat{\mathbf{u}}=\left[1+\mathcal{O}\left(Ro^3\right)\right]\left[\hat{\nabla}\cdot\hat{\mathbf{u}}+\frac{L}{a}\tan\phi_0\left(\hat{y}\frac{\partial\hat{u}}{\partial\hat{x}}-\hat{v}\right)+\mathcal{O}\left(Ro^2\right)\right]
\tag{6.56}
$$

where

$$
\hat{\nabla}\cdot\hat{\mathbf{u}}=\frac{\partial\hat{u}}{\partial\hat{x}}+\frac{\partial\hat{v}}{\partial\hat{y}}
\tag{6.57}
$$

Hence (6.38) becomes

$$
\begin{aligned}
0=&\,\mathcal{O}\left(Ro^2\right)\\
&+\left(1+Ro\frac{L^2}{L_d^2}\right)\left\{\left[1+\mathcal{O}\left(Ro^3\right)\right]\left[\hat{\nabla}\cdot\hat{\mathbf{u}}+\frac{L}{a}\tan\phi_0\left(\hat{y}\frac{\partial\hat{u}}{\partial\hat{x}}-\hat{v}\right)+\mathcal{O}\left(Ro^2\right)\right]\right.\\
&\left.+\frac{1}{\overline{\rho}}\frac{\partial}{\partial\hat{z}}\left(\overline{\rho}\hat{w}\right)+\mathcal{O}\left(Ro^3\right)\right\}
\end{aligned}
\tag{6.58}
$$

Scale-Asymptotic Treatment

As in the shallow-water case we now expand all fields in the Rossby number:

$$
\begin{pmatrix}\hat{\mathbf{v}}\\ \hat{\rho}\\ \hat{p}\end{pmatrix}=\sum_{i=0}^{\infty}Ro^i\begin{pmatrix}\hat{\mathbf{v}}_i\\ \hat{\rho}_i\\ \hat{p}_i\end{pmatrix}
\tag{6.59}
$$

This is inserted into (6.52–6.54) and (6.58), and then all is sorted in terms of powers of Ro.

The leading order of the horizontal-momentum equations is $\mathcal{O}(1)$. It yields

$$\hat{v}_0 = \frac{1}{\hat{f}_0}\frac{\partial \hat{p}_0}{\partial \hat{x}} \tag{6.60}$$

$$\hat{u}_0 = -\frac{1}{\hat{f}_0}\frac{\partial \hat{p}_0}{\partial \hat{y}} \tag{6.61}$$

This is the geostrophic equilibrium of the horizontal wind to leading order. A consequence is that the latter has no divergence:

$$\hat{\nabla} \cdot \hat{\mathbf{u}}_0 = 0 \tag{6.62}$$

The leading order of the vertical-momentum equation is $\mathcal{O}(Ro^{-2})$. One obtains

$$\hat{\rho}_0 + \frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}(\overline{\rho}\hat{p}_0) = 0 \tag{6.63}$$

This means that the leading-order pressure and density fluctuations are in hydrostatic equilibrium. The leading order of the continuity equation is $\mathcal{O}(1)$, yielding

$$\hat{\nabla} \cdot \hat{\mathbf{u}}_0 + \frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}(\overline{\rho}\hat{w}_0) = 0 \tag{6.64}$$

which is, because of (6.62), equivalent to

$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}(\overline{\rho}\hat{w}_0) = 0 \tag{6.65}$$

Since $\overline{\rho} \to 0$ for $z \to \infty$, a diverging vertical wind at infinity can only be avoided by having everywhere

$$\hat{w}_0 = 0 \tag{6.66}$$

Hence, the vertical wind is not only weaker than the horizontal wind by a factor H/L but even by a factor $Ro\, H/L$.

Now turning to the next order $\mathcal{O}(Ro)$ of the horizontal-momentum equations, we obtain

$$\frac{D_0}{D\hat{t}}\hat{u}_0 - \hat{f}_0\hat{v}_1 - \hat{\beta}\hat{y}\hat{v}_0 = -\frac{\partial \hat{p}_1}{\partial \hat{x}} - \frac{L/a}{Ro}\tan\phi_0\, \hat{y}\frac{\partial \hat{p}_0}{\partial \hat{x}} \tag{6.67}$$

$$\frac{D_0}{D\hat{t}}\hat{v}_0 + \hat{f}_0\hat{u}_1 + \hat{\beta}\hat{y}\hat{u}_0 = -\frac{\partial \hat{p}_1}{\partial \hat{y}} \tag{6.68}$$

where

$$\frac{D_0}{D\hat{t}} = \frac{\partial}{\partial \hat{t}} + \hat{u}_0\frac{\partial}{\partial \hat{x}} + \hat{v}_0\frac{\partial}{\partial \hat{y}} \tag{6.69}$$

is the non-dimensional form of the quasigeostrophic material derivative. Via ∂ (6.68) $/\partial x - \partial$ (6.67) ∂y we obtain the equation

$$\frac{D_0\hat{\zeta}_0}{D\hat{t}} + \hat{v}_0\frac{\partial}{\partial\hat{y}}\hat{\beta}\hat{y} = -\hat{f}_0\left[\hat{\nabla}\cdot\hat{\mathbf{u}}_1 + \frac{L/a}{Ro}\tan\phi_0\left(\hat{y}\frac{\partial\hat{u}_0}{\partial\hat{x}} - \hat{v}_0\right)\right] \tag{6.70}$$

for the non-dimensional quasigeostrophic vorticity

$$\hat{\zeta}_0 = \frac{\partial\hat{v}_0}{\partial\hat{x}} - \frac{\partial\hat{u}_0}{\partial\hat{y}} = \frac{1}{\hat{f}_0}\left(\frac{\partial^2\hat{p}_0}{\partial\hat{x}^2} + \frac{\partial^2\hat{p}_0}{\partial\hat{y}^2}\right) \tag{6.71}$$

where again (6.62) has been used, and, also resulting therefrom,

$$\frac{\partial\hat{\mathbf{u}}_0}{\partial\hat{x}}\cdot\hat{\nabla}\hat{v}_0 = 0 \tag{6.72}$$

$$\frac{\partial\hat{\mathbf{u}}_0}{\partial\hat{y}}\cdot\hat{\nabla}\hat{u}_0 = 0 \tag{6.73}$$

On the other hand, the $\mathcal{O}(Ro)$ of the continuity equation yields

$$0 = \hat{\nabla}\cdot\hat{\mathbf{u}}_1 + \frac{L/a}{Ro}\tan\phi_0\left(\hat{y}\frac{\partial\hat{u}_0}{\partial\hat{x}} - \hat{v}_0\right) + \frac{1}{\overline{\rho}}\frac{\partial}{\partial\hat{z}}(\overline{\rho}\hat{w}_1) \tag{6.74}$$

so that we obtain the quasigeostrophic vorticity equation

$$\boxed{\frac{D_0}{D\hat{t}}\left(\hat{\zeta}_0 + \hat{\beta}\hat{y}\right) = \frac{1}{\overline{\rho}}\frac{\partial}{\partial\hat{z}}\left(\overline{\rho}\hat{w}_1\right)} \tag{6.75}$$

This equation is not closed. All terms on the left-hand side can be calculated from $\hat{p}_0$ and its derivatives. A connection to $\hat{w}_1$, however, is not yet discernible. For further progress we now turn to the thermodynamics, in the form of the entropy equation.

6.1.2 Analysis of the Entropy Equation

We first consider potential temperature. By way of (6.28), (5.70), and (5.85) the pressure can be written as

$$p = \overline{p}\left(1 + \frac{gH\overline{\rho}}{\overline{p}}Ro\frac{L^2}{L_d^2}\hat{p}\right) \tag{6.76}$$

Since the reference atmosphere is hydrostatic and H approximately corresponds to the hydrostatic scale height, one has $gH\overline{\rho}/\overline{p} = \mathcal{O}(1)$ so that the second term in (6.76) is small. Expressing the temperature in the definition (2.91) of the potential temperature via the equation of state (2.3) in terms of pressure and density we obtain

$$\theta = \frac{p_{00}}{R\rho}\left(\frac{p_{00}}{p}\right)^{R/c_p-1} \tag{6.77}$$

which yields, with the help of (6.1), (6.6), (6.33), and (6.76)

$$\theta = \overline{\theta}\frac{\left(1 + \frac{\overline{\rho}}{\overline{p}} g H Ro \frac{L^2}{L_d^2} \hat{p}\right)^{1-R/c_p}}{\left(1 + Ro\frac{L^2}{L_d^2}\hat{\rho}\right)} \tag{6.78}$$

where

$$\overline{\theta} = \frac{p_{00}}{R\overline{\rho}}\left(\frac{p_{00}}{\overline{p}}\right)^{R/c_p - 1} \tag{6.79}$$

is the reference-atmosphere potential temperature. Due to (6.40) this leads to

$$\frac{\theta}{\overline{\theta}} = 1 - Ro\frac{L^2}{L_d^2}\hat{\rho} + \left(1 - \frac{R}{c_p}\right)\frac{gH\overline{\rho}}{\overline{p}} Ro\frac{L^2}{L_d^2}\hat{p} + \mathcal{O}\left(Ro^4\right) \tag{6.80}$$

The order of magnitude of the second but last term in (6.80) needs closer consideration. First, due to the hydrostatic equilibrium (6.3) of the reference atmosphere, and (6.10), one has

$$\frac{1}{H}\frac{d\overline{p}}{d\hat{z}} = -g\overline{\rho} \tag{6.81}$$

so that

$$\left(1 - \frac{R}{c_p}\right)\frac{gH\overline{\rho}}{\overline{p}} = \left(\frac{R}{c_p} - 1\right)\frac{1}{\overline{p}}\frac{d\overline{p}}{d\hat{z}} \tag{6.82}$$

Moreover from (6.79) follows

$$\frac{1}{\overline{\theta}}\frac{d\overline{\theta}}{d\hat{z}} = -\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{d\hat{z}} - \left(\frac{R}{c_p} - 1\right)\frac{1}{\overline{p}}\frac{d\overline{p}}{d\hat{z}} \tag{6.83}$$

so that (6.82) becomes

$$\left(1 - \frac{R}{c_p}\right)\frac{\overline{\rho} g H}{\overline{p}} = -\frac{1}{\overline{\theta}}\frac{d\overline{\theta}}{d\hat{z}} - \frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{d\hat{z}} \tag{6.84}$$

Since H corresponds to the atmospheric scale height one has

$$\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{d\hat{z}} = \mathcal{O}(1) \tag{6.85}$$

On the other hand, due to (2.143) and (6.10)

$$\frac{1}{\overline{\theta}}\frac{d\overline{\theta}}{d\hat{z}} = \frac{HN^2}{g} \tag{6.86}$$

holds. In the troposphere typically $N^2 = \mathcal{O}(10^{-4}s^{-2})$ so that

$$\frac{HN^2}{g} = \mathcal{O}\left(\frac{10^4 \cdot 10^{-4}}{10}\right) = \mathcal{O}\left(Ro\right) \tag{6.87}$$

Thus one can write

$$\boxed{\frac{1}{\overline{\theta}}\frac{d\overline{\theta}}{d\hat{z}} = Ro\hat{N}^2} \tag{6.88}$$

where

$$\hat{N}^2 = \frac{HN^2}{gRo} = \mathcal{O}\left(1\right) \tag{6.89}$$

Using (6.84) and (6.88) one can finally rewrite (6.80) as

$$\boxed{\theta = \overline{\theta}\left[1 - Ro\frac{L^2}{L_d^2}\left(\hat{\rho} + \frac{\hat{p}}{\overline{\rho}}\frac{d\overline{\rho}}{d\hat{z}}\right) + \mathcal{O}\left(Ro^3\right)\right]} \tag{6.90}$$

We therefore write

$$\theta = \overline{\theta}\left(1 + Ro\frac{L^2}{L_d^2}\hat{\theta}\right) \tag{6.91}$$

and expand $\hat{\theta}$ in terms of Ro:

$$\hat{\theta} = \sum_{i=0}^{\infty} Ro^i \hat{\theta}_i \tag{6.92}$$

The comparison with (6.90) yields

$$\hat{\theta}_0 = -\hat{\rho}_0 - \frac{\hat{p}_0}{\overline{\rho}}\frac{d\overline{\rho}}{d\hat{z}} \tag{6.93}$$

which gives together with (6.63)

$$\boxed{\hat{\theta}_0 = \frac{\partial \hat{p}_0}{\partial \hat{z}}} \tag{6.94}$$

To leading order the deviation of potential temperature from its reference-atmosphere value is thus determined by $\hat{p}_0$. A consequence of this is the *thermal-wind relation*, since the vertical derivatives of (6.60) and (6.61) yield, using (6.94) and $\hat{f}_0 = 1$,

$$\frac{\partial \hat{u}_0}{\partial \hat{z}} = -\frac{\partial \hat{\theta}_0}{\partial \hat{y}} \tag{6.95}$$

$$\frac{\partial \hat{v}_0}{\partial \hat{z}} = \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \tag{6.96}$$

As a consequence of hydrostatic and geostrophic equilibrium, the horizontal potential-temperature gradients are thus equivalent to vertical gradients of the horizontal wind.

Now consider the entropy equation (2.126) *without friction and heat conduction*. In analogous manner to the treatment of the momentum and continuity equations, also using (6.91), we first non-dimensionalize the left-hand side. The result is

$$\frac{U}{L}\frac{\tilde{D}}{D\hat{t}}\left[\overline{\theta}\left(1+Ro\frac{L^2}{L_d^2}\hat{\theta}\right)\right]=\frac{q\theta}{c_pT} \tag{6.97}$$

Since $\overline{\theta}$ is only altitude-dependent one obtains, also with the help of (6.88),

$$Ro\frac{\tilde{D}\hat{\theta}}{D\hat{t}}+\left(1+Ro\frac{L^2}{L_d^2}\hat{\theta}\right)\frac{Ro\hat{N}^2}{L^2/L_d^2}\hat{w}=\frac{q}{c_pT}\frac{L_d^2}{UL}\frac{\theta}{\overline{\overline{\theta}}} \tag{6.98}$$

Due to (6.46) and (6.47) the material derivative is given to leading order by the quasigeostrophic material derivative (6.69). Moreover, the vertical wind becomes to leading order $Ro\hat{w}_1$. Further resorting to (6.40) one sees that the leading order of the left-hand side of this equation is $\mathcal{O}(Ro)$ so that we write for consistency

$$\frac{q}{c_pT}\frac{L_d^2}{UL}\frac{\theta}{\overline{\overline{\theta}}}=Ro\hat{Q} \tag{6.99}$$

The leading order $\mathcal{O}(Ro)$ of the total equation is thus

$$\frac{D_0\hat{\theta}_0}{D\hat{t}}+S\hat{w}_1=\hat{Q} \tag{6.100}$$

where

$$S=Ro\frac{L_d^2}{L^2}\hat{N}^2=\mathcal{O}(1) \tag{6.101}$$

is a stability parameter. Now we have succeeded since (6.100) can be solved for $\hat{w}_1$, yielding

$$\boxed{\hat{w}_1=\frac{1}{S}\left(\hat{Q}-\frac{D_0\hat{\theta}_0}{D\hat{t}}\right)} \tag{6.102}$$

6.1.3 Quasigeostrophic Potential Vorticity in the Stratified Atmosphere

The vertical wind from the estimate (6.102) above is now used in the vorticity equation (6.75). One has

$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial\hat{z}}(\overline{\rho}\hat{w}_1)=\frac{1}{\overline{\rho}}\frac{\partial}{\partial\hat{z}}\left(\overline{\rho}\frac{\hat{Q}}{S}\right)-\frac{1}{\overline{\rho}}\frac{\partial}{\partial\hat{z}}\left(\frac{\overline{\rho}}{S}\frac{D_0\hat{\theta}_0}{D\hat{t}}\right) \tag{6.103}$$

Since $\overline{\rho}/S$ only depends on $\hat{z}$, the second term is, with the definition (6.69) of the quasigeostrophic material derivative,

$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}\left(\frac{\overline{\rho}}{S}\frac{D_0\hat{\theta}_0}{D\hat{t}}\right) = \frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}\left[\frac{D_0}{D\hat{t}}\left(\frac{\overline{\rho}}{S}\hat{\theta}_0\right)\right]$$
$$= \frac{D_0}{D\hat{t}}\left[\frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}\left(\frac{\overline{\rho}}{S}\hat{\theta}_0\right)\right] + \frac{1}{S}\left(\frac{\partial \hat{u}_0}{\partial \hat{z}}\frac{\partial \hat{\theta}_0}{\partial \hat{x}} + \frac{\partial \hat{v}_0}{\partial \hat{z}}\frac{\partial \hat{\theta}_0}{\partial \hat{y}}\right)$$
$$= \frac{D_0}{D\hat{t}}\left[\frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}\left(\frac{\overline{\rho}}{S}\hat{\theta}_0\right)\right] \tag{6.104}$$

In the last step the thermal-wind relations (6.95) and (6.96) have been used. Thus one has

$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}\left(\overline{\rho}\hat{w}_1\right) = \frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}\left(\frac{\overline{\rho}}{S}\hat{Q}\right) - \frac{D_0}{D\hat{t}}\left[\frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}\left(\frac{\overline{\rho}}{S}\hat{\theta}_0\right)\right] \tag{6.105}$$

Inserting this into (6.75) finally yields the desired non-dimensional conservation equation

$$\frac{D_0}{D\hat{t}}\left[\hat{\zeta}_0 + \hat{\beta}\hat{y} + \frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}\left(\frac{\overline{\rho}}{S}\hat{\theta}_0\right)\right] = \frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}\left(\frac{\overline{\rho}}{S}\hat{Q}\right) \tag{6.106}$$

Now also defining the non-dimensional streamfunction

$$\hat{\psi} = \hat{p}_0 \tag{6.107}$$

so that due to (6.94)

$$\hat{\theta}_0 = \frac{\partial \hat{\psi}}{\partial \hat{z}} \tag{6.108}$$

and, because of the geostrophy (6.60) and (6.61) of the horizontal wind and $\hat{f}_0 = 1$, one also has

$$\hat{u}_0 = -\frac{\partial \hat{\psi}}{\partial \hat{y}} \tag{6.109}$$
$$\hat{v}_0 = \frac{\partial \hat{\psi}}{\partial \hat{x}} \tag{6.110}$$

the conservation equation becomes

$$\left(\frac{\partial}{\partial \hat{t}} - \frac{\partial \hat{\psi}}{\partial \hat{y}}\frac{\partial}{\partial \hat{x}} + \frac{\partial \hat{\psi}}{\partial \hat{x}}\frac{\partial}{\partial \hat{y}}\right)\left[\frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} + \hat{\beta}\hat{y} + \frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}\left(\frac{\overline{\rho}}{S}\frac{\partial \hat{\psi}}{\partial \hat{z}}\right)\right] = \frac{1}{\overline{\rho}}\frac{\partial}{\partial \hat{z}}\left(\frac{\overline{\rho}}{S}\hat{Q}\right) \tag{6.111}$$

For practical use we now re-introduce the dimensions. First we define for the streamfunction

$$\psi = UL\hat{\psi} \tag{6.112}$$

so that the geostrophic wind

$$\mathbf{u}_g = \begin{pmatrix} u_g \\ v_g \end{pmatrix} = U \begin{pmatrix} \hat{u}_0 \\ \hat{v}_0 \end{pmatrix} \tag{6.113}$$

can be calculated from this, using (6.109) and (6.110), as

$$u_g = -\frac{\partial \psi}{\partial y} \tag{6.114}$$

$$v_g = \frac{\partial \psi}{\partial x} \tag{6.115}$$

Moreover one has on the β-plane

$$\begin{pmatrix} x \\ y \end{pmatrix} = L \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \tag{6.116}$$

Via the definition (5.85) of the external Rossby deformation radius and (6.89) one also finds that

$$S = \frac{L_{di}^2}{L^2} \tag{6.117}$$

where

$$L_{di} = \frac{HN}{f_0} \tag{6.118}$$

is the *internal Rossby deformation radius*. Finally also using (6.9) for the redimensionalization of time, (6.10) for that of $\hat{z}$, and taking (6.24) and (6.25) into consideration, one finally obtains the conservation equation

$$\frac{D_g \pi}{Dt} = \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \left(\overline{\rho} \frac{f_0 g}{N^2} \frac{q}{c_p \overline{T}} \right) \tag{6.119}$$

for the quasigeotrophic potential vorticity

$$\pi = \nabla_h^2 \psi + f_0 + \beta y + \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \left(\overline{\rho} \frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right) \tag{6.120}$$

where

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} = \frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \tag{6.121}$$

is the quasigeostrophic material derivative. Here we have used the approximation, at good accuracy, that $\theta/\overline{\theta} T = \overline{T}$. Furthermore one has

$$\nabla_h^2 \psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi \tag{6.122}$$

Without heating (and friction and heat conduction) the *quasigeostrophic potential vorticity* π is thus conserved. Most importantly, the conservation equation is a prognostic equation for the streamfunction from which all other fields can be determined. The horizontal wind follows from geostrophy. The vertical wind is via (6.94), (6.102), and (6.107)

$$w = W Ro\, \hat{w}_1 = Ro \frac{W}{S}\left(\hat{Q} - \frac{D_0 \hat{\theta}_0}{D\hat{t}}\right) = Ro \frac{W}{S}\left(\hat{Q} - \frac{D_0}{D\hat{t}}\frac{\partial \hat{\psi}}{\partial \hat{z}}\right) \tag{6.123}$$

With the help of (6.16), (5.85), (6.99), (6.117), (6.118) and all the redimensionalization steps having led to (6.119) one obtains from this

$$w = \frac{g}{N^2}\frac{q}{c_p \overline{T}} - \frac{f_0}{N^2}\frac{D_g}{Dt}\frac{\partial \psi}{\partial z} \tag{6.124}$$

Pressure is obtained via (6.29), (6.107), and (6.112), yielding

$$p = \overline{p} + f_0 \overline{\rho} \psi \tag{6.125}$$

Finally potential temperature is, via (6.91), (5.85), (6.107), (6.112), and (6.10)

$$\theta = \overline{\theta}\left(1 + \frac{f_0}{g}\frac{\partial \psi}{\partial z}\right) \tag{6.126}$$

while it is left as an exercise to the interested reader to show that

$$\rho = \overline{\rho}\left[1 - \frac{f_0}{g}\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\psi\right)\right] \tag{6.127}$$

In a summary all fields, obtained from the streamfunction, are

$$u = -\frac{\partial \psi}{\partial y} \tag{6.128}$$

$$v = \frac{\partial \psi}{\partial x} \tag{6.129}$$

$$w = \frac{g}{N^2}\frac{q}{c_p \overline{T}} - \frac{f_0}{N^2}\frac{D_g}{Dt}\frac{\partial \psi}{\partial z} \tag{6.130}$$

$$p = \overline{p} + f_0 \overline{\rho} \psi \tag{6.131}$$

$$\rho = \overline{\rho}\left[1 - \frac{f_0}{g}\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\psi\right)\right] \tag{6.132}$$

$$\theta = \overline{\theta}\left(1 + \frac{f_0}{g}\frac{\partial \psi}{\partial z}\right) \tag{6.133}$$

In an analogous manner one finally also finds that the dimensional form of the thermal-wind relations (6.95) and (6.96) is

$$\frac{\partial u}{\partial z} = -\frac{g}{f_0}\frac{\partial}{\partial y}\left(\frac{\theta'}{\overline{\theta}}\right) \tag{6.134}$$

$$\frac{\partial v}{\partial z} = \frac{g}{f_0}\frac{\partial}{\partial x}\left(\frac{\theta'}{\overline{\theta}}\right) \tag{6.135}$$

where $\theta' = \theta - \overline{\theta}$ is the deviation of potential temperature from that of the reference atmosphere.

6.1.4 The Relationship with General Potential Vorticity

Closer inspection shows that the quasigeostrophic potential vorticity is *not* simply an approximation of Ertel's potential vorticity

$$\Pi = \frac{\boldsymbol{\omega}_a}{\rho} \cdot \nabla\theta \tag{6.136}$$

in the limit of synoptic scaling. Rather its conservation (in the absence of heating, friction, and heat conduction) follows from scale-asymptotic analyses of the conservation equations for general potential vorticity and potential temperature. Beyond this also the result from the continuity equation finds application that to leading order the quasigeostrophic flow is horizontal, which again is a consequence of the vanishing of its horizontal divergence. This shall be demonstrated here.

First we decompose the absolute vorticity

$$\boldsymbol{\omega}_a = \boldsymbol{\omega} + 2\boldsymbol{\Omega} \tag{6.137}$$

via (3.29) and (4.97–4.99) into its spherical-coordinate components so that

$$\Pi = \frac{\omega_{a\lambda}}{\rho}\frac{1}{r\cos\phi}\frac{\partial\theta}{\partial\lambda} + \frac{\omega_{a\phi}}{\rho}\frac{1}{r}\frac{\partial\theta}{\partial\phi} + \frac{\omega_{ar}}{\rho}\frac{\partial\theta}{\partial r} \tag{6.138}$$

where

$$\omega_{a\lambda} = \frac{1}{r}\frac{\partial w}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rv) \tag{6.139}$$

$$\omega_{a\phi} = 2\Omega\cos\phi + \frac{1}{r}\frac{\partial}{\partial r}(ru) - \frac{1}{r\cos\phi}\frac{\partial w}{\partial\lambda} \tag{6.140}$$

$$\omega_{ar} = 2\Omega\sin\phi + \frac{1}{r\cos\phi}\frac{\partial v}{\partial\lambda} - \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\, u) \tag{6.141}$$

By way of the scaling steps and results from Sects. 6.1.1 and 6.1.2 we now analyze the three contributing terms. They are

$$\frac{\omega_{a\lambda}}{\rho}\frac{1}{r\cos\phi}\frac{\partial\theta}{\partial\lambda} = \frac{Ro^2\dfrac{L^2}{L_d^2}\dfrac{\overline{\theta}}{\overline{\rho}}\dfrac{f_0}{H}}{1+Ro\dfrac{L^2}{L_d^2}\hat{\rho}}\frac{\partial\hat{\theta}/\partial\hat{\lambda}}{\left(1+\dfrac{H}{a}\hat{z}\right)\cos\phi}$$
$$\times\left\{\frac{H^2}{L^2}\frac{\partial\hat{w}/\partial\hat{\phi}}{1+\dfrac{H}{a}\hat{z}} - \frac{1}{1+\dfrac{H}{a}\hat{z}}\frac{\partial}{\partial\hat{z}}\left[\left(1+\frac{H}{a}\hat{z}\right)\hat{v}\right]\right\}$$
$$= \mathcal{O}\left(Ro^2\frac{L^2}{L_d^2}\frac{\overline{\theta}}{\overline{\rho}}\frac{f_0}{H}\right) = \mathcal{O}\left(Ro^3\frac{\overline{\theta}}{\overline{\rho}}\frac{f_0}{H}\right) \tag{6.142}$$

$$\frac{\omega_{a\phi}}{\rho}\frac{1}{r}\frac{\partial\theta}{\partial\phi} = \frac{Ro^2\dfrac{L^2}{L_d^2}\dfrac{\overline{\theta}}{\overline{\rho}}\dfrac{f_0}{H}}{1+Ro\dfrac{L^2}{L_d^2}\hat{\rho}}\frac{\partial\hat{\theta}/\partial\hat{\phi}}{\left(1+\dfrac{H}{a}\hat{z}\right)}$$
$$\times\left\{\frac{Ha}{L^2}\hat{\beta} + \frac{1}{1+\dfrac{H}{a}\hat{z}}\frac{\partial}{\partial\hat{z}}\left[\left(1+\frac{H}{a}\hat{z}\right)\hat{u}\right] - \frac{H^2}{L^2}\frac{\partial\hat{w}/\partial\hat{\lambda}}{\left(1+\dfrac{H}{a}\hat{z}\right)\cos\phi}\right\}$$
$$= \mathcal{O}\left(Ro^2\frac{L^2}{L_d^2}\frac{\overline{\theta}}{\overline{\rho}}\frac{f_0}{H}\right) = \mathcal{O}\left(Ro^3\frac{\overline{\theta}}{\overline{\rho}}\frac{f_0}{H}\right) \tag{6.143}$$

$$\frac{\omega_{ar}}{\rho}\frac{\partial\theta}{\partial r} = \frac{\dfrac{\overline{\theta}}{\overline{\rho}}\dfrac{f_0}{H}}{1+Ro\dfrac{L^2}{L_d^2}\hat{\rho}}\left[\left(1+Ro\frac{L^2}{L_d^2}\hat{\rho}\right)Ro\hat{N}^2 + Ro\frac{L^2}{L_d^2}\frac{\partial\hat{\theta}}{\partial\hat{z}}\right]$$
$$\times\left\{\hat{f} + \frac{Ro}{1+\dfrac{H}{a}\hat{z}}\left[\frac{1}{\cos\phi}\frac{\partial\hat{v}}{\partial\hat{\lambda}} - \frac{1}{\cos\phi}\frac{\partial}{\partial\hat{\phi}}\left(\cos\phi\hat{u}\right)\right]\right\}$$
$$= \frac{\overline{\theta}}{\overline{\rho}}\frac{f_0}{H}\left[1+\mathcal{O}\left(Ro^2\right)\right]$$
$$\times\left\{Ro\hat{f}_0\hat{N}^2 + Ro^2\left[\hat{N}^2\left(\hat{\beta}\hat{y} + \frac{\partial\hat{v}}{\partial\hat{x}} - \frac{\partial\hat{u}}{\partial\hat{y}}\right) + \frac{L^2/L_d^2}{Ro}\hat{f}_0\frac{\partial\hat{\theta}}{\partial\hat{z}}\right]\right.$$
$$\left. +\mathcal{O}\left(Ro^3\right)\right\} \tag{6.144}$$

so that potential vorticity takes the asymptotic form

$$\Pi = \frac{\overline{\theta}}{\overline{\rho}} \frac{f_0}{H} \left\{ Ro \hat{f}_0 \hat{N}^2 + Ro^2 \left[\hat{N}^2 \left(\hat{\beta} \hat{y} + \frac{\partial \hat{v}}{\partial \hat{x}} - \frac{\partial \hat{u}}{\partial \hat{y}} \right) + \frac{L^2/L_d^2}{Ro} \hat{f}_0 \frac{\partial \hat{\theta}}{\partial \hat{z}} \right] + \mathcal{O}\left(Ro^3\right) \right\} \tag{6.145}$$

In the application of the material derivative we use (6.46) and $\hat{w}_0 = 0$ so that

$$\frac{D}{Dt} = \frac{U}{L} \left\{ \frac{D_0}{D\hat{t}} + Ro \left[\left(\frac{L/a}{Ro} \tan \phi_0 \, \hat{y} \, \hat{u}_0 + \hat{u}_1 \right) \frac{\partial}{\partial \hat{x}} + \hat{v}_1 \frac{\partial}{\partial \hat{y}} + \hat{w}_1 \frac{\partial}{\partial \hat{z}} \right] + \mathcal{O}\left(Ro^2\right) \right\} \tag{6.146}$$

and hence

$$\begin{aligned} 0 &= \frac{D\Pi}{Dt} \\ &= \frac{U}{L} \frac{\overline{\theta}}{\overline{\rho}} \frac{f_0}{H} \hat{N}^2 \\ &\quad \times \left\{ Ro^2 \left[\frac{D_0}{D\hat{t}} \left(\frac{\partial \hat{v}_0}{\partial \hat{x}} - \frac{\partial \hat{u}_0}{\partial y} + \hat{\beta} \hat{y} + \frac{\hat{f}_0}{S} \frac{\partial \hat{\theta}_0}{\partial \hat{z}} \right) + \frac{\overline{\rho}}{\overline{\theta}} \frac{\hat{w}_1}{\hat{N}^2} \frac{d}{d\hat{z}} \left(\hat{f}_0 \frac{\overline{\theta}}{\overline{\rho}} \hat{N}^2 \right) \right] \right. \\ &\quad \left. + \mathcal{O}\left(Ro^3\right) \right\} \end{aligned} \tag{6.147}$$

Herein one has, due to (6.88) and (6.101),

$$\frac{\overline{\rho}}{\overline{\theta}} \frac{\hat{w}_1}{\hat{N}^2} \frac{d}{d\hat{z}} \left(\hat{f}_0 \frac{\overline{\theta}}{\overline{\rho}} \hat{N}^2 \right) = \frac{\overline{\rho}}{\overline{\theta}} \frac{\hat{w}_1}{S} \frac{d}{d\hat{z}} \left(\hat{f}_0 \frac{\overline{\theta}}{\overline{\rho}} S \right) = \hat{f}_0 \frac{\hat{w}_1}{S} \left[\overline{\rho} \frac{d}{d\hat{z}} \left(\frac{S}{\overline{\rho}} \right) + \mathcal{O}(Ro) \right] \tag{6.148}$$

so that to leading order

$$0 = \frac{D_0}{D\hat{t}} \left(\hat{\zeta}_0 + \hat{\beta} \hat{y} + \frac{\hat{f}_0}{S} \frac{\partial \hat{\theta}_0}{\partial \hat{z}} \right) + \hat{f}_0 \hat{w}_1 \frac{\overline{\rho}}{S} \frac{d}{d\hat{z}} \left(\frac{S}{\overline{\rho}} \right) \tag{6.149}$$

Now we use conservation of potential temperature in the adiabatic case $\hat{Q} = 0$. Inserting the corresponding result (6.102) for $\hat{w}_1$ leads to the conservation equation

$$0 = \frac{D_0}{D\hat{t}} \left[\hat{\zeta}_0 + \hat{\beta} \hat{y} + \frac{1}{\overline{\rho}} \frac{\partial}{\partial \hat{z}} \left(\frac{\overline{\rho}}{S} \hat{\theta}_0 \right) \right] \tag{6.150}$$

which agrees with (6.106) in the case $\hat{Q} = 0$.

6.1.5 Quasigeostrophic Theory in Pressure Coordinates

Beginning with the primitive equations we can use an analogous scale asymptotics as above for deriving also in pressure coordinates a potential vorticity and its conservation equation. Instead of this rather formal procedure, however, we choose here a heuristic approach. This somewhat better illuminates how the various basic assumptions of quasigeostrophic theory act together. For simplicity we right away begin with the primitive equations on the β-plane.

The two momentum equations are

$$\frac{Du}{Dt} - (f_0 + \beta y)v = -\frac{\partial \Phi}{\partial x} \tag{6.151}$$

$$\frac{Dv}{Dt} + (f_0 + \beta y)u = -\frac{\partial \Phi}{\partial y} \tag{6.152}$$

To leading order the Coriolis effect without β-term dominates in these together with the pressure-gradient acceleration so that the horizontal wind is approximately in geostrophic equilibrium:

$$u \approx u_g = -\frac{1}{f_0}\frac{\partial \Phi_g}{\partial y} \tag{6.153}$$

$$v \approx v_g = \frac{1}{f_0}\frac{\partial \Phi_g}{\partial x} \tag{6.154}$$

Here we have decomposed the geopotential $\Phi = \overline{\Phi}(p) + \Phi_g + \Phi_a$ into the reference-atmosphere part, and a fluctuating remainder that is dominated by a part Φ_g participating in the geostrophic equilibrium, and has an ageostrophic remainder Φ_a. The geostrophic horizontal wind thus also is non-divergent so that

$$\nabla \cdot \mathbf{u}_g = 0 \tag{6.155}$$

We now decompose the wind into its dominant geostrophic part and the ageostrophic rest:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}_g + \begin{pmatrix} u \\ v \\ w \end{pmatrix}_a \tag{6.156}$$

The continuity Eq. (3.92) is to leading order:

$$\nabla \cdot \mathbf{u}_g + \frac{\partial \omega_g}{\partial p} = 0 \tag{6.157}$$

Due to the non-divergence of the geostrophic wind one thus has

$$\frac{\partial \omega_g}{\partial p} = 0 \tag{6.158}$$

The upper boundary condition is $\omega\,(p \to 0) = 0$, or to leading order $\omega_g\,(p \to 0) = 0$, yielding

$$\omega_g = 0 \tag{6.159}$$

With these estimates the material derivative becomes to leading order

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla + \omega \frac{\partial}{\partial p} \approx \frac{D_g}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla \tag{6.160}$$

Now we turn to the momentum equations (6.151) and (6.152), use there the horizontal-wind decomposition (6.156) and the approximation (6.160) of the material derivative. Further taking the geostrophic equilibrium (6.153–6.154) into account and neglecting $\beta y \mathbf{u}_a$ in comparison to $\beta y \mathbf{u}_g$ we obtain

$$\frac{D_g u_g}{Dt} - f_0 v_a - \beta y v_g = -\frac{\partial \Phi_a}{\partial x} \tag{6.161}$$

$$\frac{D_g v_g}{Dt} + f_0 u_a + \beta y u_g = -\frac{\partial \Phi_a}{\partial y} \tag{6.162}$$

$\partial\,(6.162)/\partial x - \partial\,(6.161)/\partial y$ yields the equation

$$\frac{D_g}{Dt}(\zeta_g + f) = -f_0 \nabla \cdot \mathbf{u}_a \tag{6.163}$$

for the quasigeostrophic vorticity

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0} \nabla_h^2 \Phi \tag{6.164}$$

The divergence of the ageostrophic wind can be obtained from the continuity equation. Due to the non-divergence of the geostrophic wind and the vanishing of the geostrophic pressure velocity the latter is

$$\nabla \cdot \mathbf{u}_a + \frac{\partial \omega_a}{\partial p} = 0 \tag{6.165}$$

so that the vorticity equation becomes

$$\boxed{\frac{D_g}{Dt}(\zeta_g + f) = f_0 \frac{\partial \omega_a}{\partial p}} \tag{6.166}$$

In complete analogy to the procedure above in Sect. 6.1.2 we now use the entropy equation (here without heating, friction, heat conduction) for an estimate of the contribution from vortex-tube stretching on the right-hand side of the vorticity equation. As there we split potential temperature into the contribution from the reference atmosphere and the rest, i.e.,

$$\theta = \overline{\theta}\,(p) + \theta' \tag{6.167}$$

and thus approximate the entropy equation as

$$\frac{D_g\theta'}{Dt} + \omega_a \frac{d\overline{\theta}}{dp} = 0 \tag{6.168}$$

This way we have

$$\omega_a = -\frac{1}{d\overline{\theta}/dp}\frac{D_g\theta'}{Dt} \tag{6.169}$$

Due to the hydrostatic equilibrium (3.91), the equation of state (2.3) and the definition (2.91) of potential temperature one has

$$\theta = -\frac{p}{R}\left(\frac{p_{00}}{p}\right)^{\frac{R}{c_p}}\frac{\partial\Phi}{\partial p} \tag{6.170}$$

Also the geopotential is split as

$$\Phi = \overline{\Phi}(p) + \Phi_g + \Phi_a \tag{6.171}$$

so that one finds

$$\overline{\theta} = -\frac{p}{R}\left(\frac{p_{00}}{p}\right)^{\frac{R}{c_p}}\frac{\partial\overline{\Phi}}{\partial p} \tag{6.172}$$

$$\theta' \approx -\frac{p}{R}\left(\frac{p_{00}}{p}\right)^{\frac{R}{c_p}}\frac{\partial\Phi_g}{\partial p} \tag{6.173}$$

Since, however, $(p_{00}/p)^{R/c_p} = \overline{\theta}/\overline{T}$, where $\overline{T}(p)$ is the reference-atmosphere temperature, one obtains after again using the equation of state

$$\boxed{\theta' = -\overline{\rho}\overline{\theta}\frac{\partial\Phi_g}{\partial p}} \tag{6.174}$$

Here $\overline{\rho}(p)$ is the reference-atmosphere density. Finally, the pressure velocity (6.169) becomes

$$\boxed{\omega = \omega_a = \frac{D_g}{Dt}\left(\frac{\overline{\rho}\overline{\theta}}{d\overline{\theta}/dp}\frac{\partial\Phi_g}{\partial p}\right)} \tag{6.175}$$

This we now insert into the vorticity equation (6.166), leading us to the conservation equation for quasigeostrophic potential vorticity. First one obtains

$$\frac{D_g}{Dt}(\zeta_g + f) = \frac{\partial}{\partial p}\frac{D_g}{Dt}\left(f_0\frac{\overline{\rho}\overline{\theta}}{d\overline{\theta}/dp}\frac{\partial\Phi_g}{\partial p}\right) \tag{6.176}$$

The pressure derivative of the geostrophic wind (6.153–6.154) leads us to the relationships

$$\frac{\partial u_g}{\partial p} = -\frac{1}{f_0}\frac{\partial^2 \Phi_g}{\partial p \partial y} \tag{6.177}$$

$$\frac{\partial v_g}{\partial p} = \frac{1}{f_0}\frac{\partial^2 \Phi_g}{\partial p \partial x} \tag{6.178}$$

by the help of which (6.176) is simplified to become

$$\frac{D_g}{Dt}(\zeta_g + f) = \frac{D_g}{Dt}\frac{\partial}{\partial p}\left(f_0 \frac{\overline{\rho}\overline{\theta}}{d\overline{\theta}/dp}\frac{\partial \Phi_g}{\partial p}\right) \tag{6.179}$$

Now we define the streamfunction

$$\boxed{\psi = \frac{\Phi_g}{f_0}} \tag{6.180}$$

and the stability parameter

$$\sigma = -\frac{1}{\overline{\rho}\overline{\theta}}\frac{d\overline{\theta}}{dp} \tag{6.181}$$

and finally obtain the desired conservation equation

$$\boxed{\frac{D_g \pi}{Dt} = 0 \qquad \pi = \nabla_h^2 \psi + f + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma}\frac{\partial \psi}{\partial p}\right)} \tag{6.182}$$

with

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} \tag{6.183}$$

Note again that it holds in the absence of friction, heating, and heat conduction. A corresponding extension, however, is possible. A characteristic value for the stability parameter in midlatitudes is $\sigma = 2 \cdot 10^{-6}\ \mathrm{m}^2/\mathrm{Pa}^2\mathrm{s}^2$. Note also that (6.177) and (6.178) can be rewritten as the thermal-wind relationships

$$\boxed{\frac{\partial u_g}{\partial p} = \frac{1}{f_0 \overline{\rho}}\frac{\partial}{\partial y}\left(\frac{\theta'}{\overline{\theta}}\right)} \tag{6.184}$$

$$\boxed{\frac{\partial v_g}{\partial p} = -\frac{1}{f_0 \overline{\rho}}\frac{\partial}{\partial x}\left(\frac{\theta'}{\overline{\theta}}\right)} \tag{6.185}$$

6.1.6 A Quasigeostrophic Two-Layer Model

The vertical structure of important synoptic-scale processes is simple enough so that it may often suffice to consider the atmosphere in the approximation of two layers. The corresponding equations will be derived here, where we limit ourselves directly to the geometry of the β-plane. Starting point is the vorticity equation (6.166) which we here write

$$\boxed{\frac{D_g}{Dt}\left(\zeta_g + f\right) = f_0 \frac{\partial \omega}{\partial p}} \tag{6.186}$$

and the entropy equation (6.168) which we express via (6.174) and (6.180) in the form

$$\boxed{\frac{D_g}{Dt}\frac{\partial \psi}{\partial p} + \frac{\sigma}{f_0}\omega = 0} \tag{6.187}$$

For a representation of the dynamics we now pick two pressure layers as in Fig. 6.1. Streamfunction and horizontal winds are defined on the two main levels 1 and 2, while the pressure velocity is defined on the side and intermediate levels at the top (t) and bottom (b) boundary and between the two layers (m). On the latter we also define potential temperature. For a discretization we now approximate the vertical derivatives by finite differences. Thus one approximates (6.186) on the upper level at $p = p_1$ as

$$\frac{D_g}{Dt}\left(\nabla_h^2 \psi_1 + f\right) = f_0 \frac{\omega_m - \omega_t}{p_m - p_t} \tag{6.188}$$

where the quasigeostrophic material derivative of an arbitrary field A in this layer is defined as

$$\frac{D_g A}{Dt} = \frac{\partial A}{\partial t} + J(\psi_1, A) \tag{6.189}$$

The Jacobi operator applied to arbitrary fields B and C is

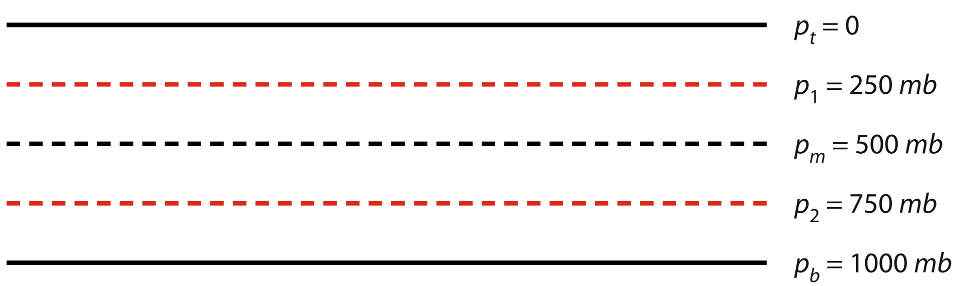

Fig. 6.1 The vertical discretization of a two-layer model

$$J(B, C) = \frac{\partial B}{\partial x}\frac{\partial C}{\partial y} - \frac{\partial B}{\partial y}\frac{\partial C}{\partial x} \tag{6.190}$$

It has the following useful properties:

$$J(A + B, C) = J(A, C) + J(B, C) \tag{6.191}$$

$$J(\alpha A, \beta B) = \alpha\beta J(A, B) \tag{6.192}$$

$$J(A, B) = -J(B, A) \tag{6.193}$$

$$J(A, A) = 0 \tag{6.194}$$

where α and β are constant factors. Furthermore as before the upper boundary condition for the pressure velocity is $\omega_t = 0$, and we define

$$p_m - p_t = p_b - p_m = \Delta p \tag{6.195}$$

so that (6.188) can also be written as

$$\frac{\partial \nabla_h^2 \psi_1}{\partial t} + J\left(\psi_1, \nabla_h^2 \psi_1 + f\right) = \frac{f_0}{\Delta p}\omega_m \tag{6.196}$$

The procedure for the lower level at $p = p_2$ is analogous. There we also neglect the effects of friction and orography so that $\omega_b = 0$, thus yielding

$$\frac{\partial \nabla_h^2 \psi_2}{\partial t} + J\left(\psi_2, \nabla_h^2 \psi_2 + f\right) = -\frac{f_0}{\Delta p}\omega_m \tag{6.197}$$

Finally we discretize for the elimination of the pressure velocity ω_m also the quasigeostrophic entropy equation (6.187) on the intermediate level at $p = p_m$. For this purpose we approximate the geostrophic horizontal winds and the streamfunction there by the corresponding arithmetic mean between the two full levels. One obtains

$$\frac{\partial}{\partial t}\left(\frac{\psi_2 - \psi_1}{\Delta p}\right) + J\left(\frac{\psi_1 + \psi_2}{2}, \frac{\psi_2 - \psi_1}{\Delta p}\right) + \frac{\sigma}{f_0}\omega_m = 0 \tag{6.198}$$

Now, however,

$$\frac{\psi_1 + \psi_2}{2} = \psi_1 + \frac{\psi_2 - \psi_1}{2} = \psi_2 - \frac{\psi_2 - \psi_1}{2} \tag{6.199}$$

so that we can derive via (6.191–6.194) from (6.198) the two identities

$$\omega_m = -\frac{\partial}{\partial t}\left[\frac{f_0}{\sigma \Delta p}(\psi_2 - \psi_1)\right] - J\left[\psi_1, \frac{f_0}{\sigma \Delta p}(\psi_2 - \psi_1)\right] \tag{6.200}$$

$$\omega_m = -\frac{\partial}{\partial t}\left[\frac{f_0}{\sigma \Delta p}(\psi_2 - \psi_1)\right] - J\left[\psi_2, \frac{f_0}{\sigma \Delta p}(\psi_2 - \psi_1)\right] \tag{6.201}$$

Inserting (6.200) into (6.196), and (6.201) into (6.197), finally yield the two conservation equations

$$0 = \frac{\partial \pi_i}{\partial t} + J(\psi_i, \pi_i) \qquad \pi_{1,2} = \nabla_h^2 \psi_1 + f \pm F(\psi_2 - \psi_1) \tag{6.202}$$

where we have

$$F = \frac{f_0^2}{\sigma \Delta p^2} \tag{6.203}$$

π_1 and π_2 are, respectively, the potential vorticity in the upper and lower layer. For later reference we also note that the contribution $\pm F\,(\psi_1 - \psi_2)$ results from the elimination of ω_m, and thus effectively from the vortex-tube stretching.

6.1.7 Summary

In the stratified atmosphere with variable density and altitude-dependent horizontal winds a comparatively closed treatment of dynamic phenomena on the *synoptic* scale can be achieved within the framework of *quasigeostrophic theory*.

- For this purpose *pressure and density* are decomposed into a *hydrostatic reference part* and *small deviations*.
- An analysis of the continuity equation shows that the *ratio between the vertical and horizontal-wind scales* cannot be larger than *the ratio between the vertical and horizontal length scales*.
- The *scale of the pressure fluctuations* follows, under the *assumption of small Rossby numbers*, directly from the horizontal-momentum equation, where the pressure gradient must be balanced by the Coriolis force.
- Since *the horizontal scale is smaller than the earth's radius by a factor of the order of the Rossby number* one can use again the approximation of the *tangential β-plane*.
- The *ratios between vertical scale and horizontal scale or earth's radius* are also *small*, more precisely they are of order Ro^2 and Ro^3. Finally we also assume that the *squares of the horizontal scale and the external Rossby deformation radius have a ratio of $\mathcal{O}(Ro)$*. This last assumption differs from quasigeostrophic shallow-water theory.
- The scale of the *density fluctuations* follows under the same assumptions from the vertical momentum equation, where the vertical pressure gradient must balance gravity.
- A *Rossby-number expansion of all dynamic fields* yields the following *leading-order* results:
  - The *horizontal wind* is in *geostrophic* balance. The *pressure fluctuations* act as *streamfunction*.
  - The *fluctuations of pressure and density* are in *hydrostatic equilibrium*.

  - The *vertical wind vanishes to leading order*. This means that the scale estimate above for the vertical wind must be corrected by a Rossby-number factor.
- The resulting quasigeostrophic *vorticity equation* contains vortex-tube stretching. The vertical wind there must be determined from the *entropy equation*. The analysis of the latter and of potential temperature yields:
  - The *vertical gradient of reference-potential temperature is sufficiently weak* so that the *potential-temperature fluctuations* are of the order Ro^2.
  - Therefore they can be determined directly from the *vertical gradient of the pressure fluctuations*, which also leads to *thermal-wind balance*.
  - The vertical wind can be determined via the entropy equation from stability, heat sources and the geostrophic material derivative of the potential-temperature fluctuations.
- Inserting this vertical wind into the vorticity equation yields the *conservation equation* for *quasigeostrophic potential vorticity*. The latter can be determined, *as all dynamic fields*, from the *streamfunction*, i.e., the pressure fluctuations!
- Stability can be written as $S = L_{di}^2/L^2$, where L_{di} is the important *internal Rossby deformation radius*.
- Similar to shallow-water theory the quasigeostrophic potential-vorticity-conservation equation can as well be derived directly from its general analogue, however under additional application of entropy equation. Quasigeostrophic potential vorticity is not simply an approximation of general potential vorticity under synoptic scaling.
- A heuristic derivation of *quasigeostrophic theory in pressure coordinates* illustrates the main steps further.
- Moreover, the formulation in pressure coordinates forms the starting point for the derivation of a quasigeostrophic *two-layer model*, reducing the dynamics in the vertical, via discretization, onto two pressure layers.

6.2 Quasigeostrophic Energetics of the Stratified Atmosphere

Energy conservation is a fundamental property both of the general equations of motion and of the primitive equations. Obviously, in the absence of friction, heating and heat conduction quasigeostrophic dynamics should have a corresponding property as well, and this section demonstrates that this is indeed the case. It also shows that energy can be exchanged between kinetic and available potential energy, the latter to be defined below, and how this can happen. For this we first consider the dynamics of the continuously stratified atmosphere and then that of the two-layer model. In both cases we neglect effects of friction, heating, and heat conduction. For simplicity we use the boundary conditions of the β-channel. The results also hold, however, in the case of periodic boundary conditions in both horizontal directions, or in the case of solid-wall boundary conditions in all horizontal directions.

6.2.1 The Continuously Stratified Atmosphere

A common variant of β-plane dynamics, for the approximation of extratropical processes, uses the boundary conditions of a zonal channel (β-*channel*), spanning the globe parallel to a latitude circle. The model volume thus is

$$
\begin{aligned}
0 &\leq x \leq L_x \\
0 &\leq y \leq L_y \\
0 &\leq z < \infty
\end{aligned}
$$

It makes sense to assume $L_x = 2\pi a \cos\phi_0$. The meridional extent is specified less clearly. One can assume, e.g., $L_y = a\pi/2$.

The Boundary Conditions

Corresponding to the picture above the boundary conditions are as follows:

- In x we assume periodicity, i.e.,

$$\psi(x) = \psi(x + L_x) \tag{6.204}$$

- The meridional boundaries are solid and impermeable so that the meridional wind vanishes there, i.e.

$$v(y = 0) = v(y = L_y) = 0 \tag{6.205}$$

  This must also hold for the geostrophic part so that

$$\left.\frac{\partial\psi}{\partial x}\right|_{y=0,L_y} = 0 \tag{6.206}$$

  Thus the streamfunction at the meridional boundaries is zonally symmetric.
- The lower boundary is solid as well. The vertical wind vanishes there, i.e., $w(z = 0) = 0$. At zero heating this also means, due to (6.130), that the quasigeostrophic material derivative of the vertical streamfunction gradient vanishes, thus

$$\left.\frac{D_g}{Dt}\frac{\partial\psi}{\partial z}\right|_{z=0} = 0 \tag{6.207}$$

- Density vanishes at infinity:

$$\overline{\rho}(z \to \infty) = 0 \tag{6.208}$$

- Via the zonal-momentum equation a further relationship at the meridional boundaries $(y = 0, L_y)$ results from the zonal and meridional boundary conditions. After subtraction of the geostrophic equilibrium the former is on the β-plane

$$\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial y} - f_0 v_a - \beta y v_g = -\frac{1}{\overline{\rho}} \frac{\partial p_a}{\partial x} \tag{6.209}$$

Here p_a is the (ageostrophic) pressure at $\mathcal{O}(Ro)$, that is not in leading-order geostrophic equilibrium with the geostrophic wind. Alternatively this relationship can be obtained by redimensionalization of (6.67), with a correspondingly generalized ageostrophic pressure. Anyway, at the meridional boundaries one has $v_g = v_a = 0$, so that

$$\frac{\partial u_g}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u_g^2}{2}\right) = -\frac{1}{\overline{\rho}} \frac{\partial p_a}{\partial x} \tag{6.210}$$

Integration in x finally yields, due to zonal periodicity of streamfunction and ageostrophic pressure,

$$\frac{\partial}{\partial t} \int_0^{L_x} dx\, u_g = -\frac{\partial}{\partial t} \int_0^{L_x} dx\, \frac{\partial \psi}{\partial y} = 0 \qquad \left(y = 0, L_y\right) \tag{6.211}$$

The Conservation Law

With the boundary conditions above one obtains the energy conservation law as follows. We multiply the conservation equation (6.119) for quasigeostrophic potential vorticity (without heating) by $-\overline{\rho}\psi$ and integrate over the total volume of the β-channel:

$$-\int_0^{\infty} dz \int_0^{L_y} dy \int_0^{L_x} dx\, \overline{\rho}\psi \left(\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla\right) \left[\nabla_h^2 \psi + f + \frac{1}{\overline{\rho}} \frac{\partial}{\partial z}\left(\overline{\rho} \frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z}\right)\right] = 0 \tag{6.212}$$

For further progress we first consider for arbitrary fields F

$$\begin{aligned}
\int_0^{L_y} dy \int_0^{L_x} dx\, \psi \mathbf{u}_g \cdot \nabla F &= \int_0^{L_y} dy \int_0^{L_x} dx\, \psi J(\psi, F) \\
&= \int_0^{L_y} dy \int_0^{L_x} dx \left(-\psi \frac{\partial \psi}{\partial y} \frac{\partial F}{\partial x} + \psi \frac{\partial \psi}{\partial x} \frac{\partial F}{\partial y}\right) \\
&= \int_0^{L_y} dy \left[-\psi \frac{\partial \psi}{\partial y} F\right]_0^{L_x} + \int_0^{L_x} dx \left[\psi \frac{\partial \psi}{\partial x} F\right]_0^{L_y}
\end{aligned} \tag{6.213}$$

Herefore we have integrated in the last step twice by parts. Due to the periodicity in x, however, the first term vanishes, and due to the impermeability of the meridional boundaries also the second. One thus has

$$\int_0^{L_y} dy \int_0^{L_x} dx\ \psi \mathbf{u}_g \cdot \nabla F = 0 \tag{6.214}$$

and (6.212) is simplified, via $\partial f/\partial t = 0$, to

$$0 = -\int_0^{\infty} dz\, \overline{\rho} \int_0^{L_y} dy \int_0^{L_x} dx\ \psi \frac{\partial}{\partial t}\left[\nabla_h^2 \psi + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right] \tag{6.215}$$

Here we again have, using partial integration,

$$\begin{aligned} -\int_0^{L_y} dy \int_0^{L_x} dx\ \psi \frac{\partial}{\partial t}\nabla_h^2 \psi = & -\int_0^{L_y} dy \left[\psi \frac{\partial}{\partial t}\frac{\partial \psi}{\partial x}\right]_0^{L_x} + \int_0^{L_y} dy \int_0^{L_x} dx \frac{\partial \psi}{\partial x}\frac{\partial}{\partial t}\frac{\partial \psi}{\partial x} \\ & -\int_0^{L_x} dx \left[\psi \frac{\partial}{\partial t}\frac{\partial \psi}{\partial y}\right]_0^{L_y} + \int_0^{L_y} dy \int_0^{L_x} dx \frac{\partial \psi}{\partial y}\frac{\partial}{\partial t}\frac{\partial \psi}{\partial y} \end{aligned} \tag{6.216}$$

The first term vanishes due to periodicity in x, as does the third, using (6.206) and (6.211). One thus has

$$-\int_0^{L_y} dy \int_0^{L_x} dx\ \psi \frac{\partial}{\partial t}\nabla_h^2 \psi = \int_0^{L_y} dy \int_0^{L_x} dx\ \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \psi}{\partial y}\right)^2\right] \tag{6.217}$$

Via partial integration in z one finally obtains

$$\begin{aligned} & -\int_0^{\infty} dz \int_0^{L_y} dy \int_0^{L_x} dx\ \overline{\rho}\psi \frac{\partial}{\partial t}\left[\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right] \\ & = -\int_0^{L_y} dy \int_0^{L_x} dx \left[\psi \frac{\partial}{\partial t}\left(\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right]_0^{\infty} + \int_0^{\infty} dz \int_0^{L_y} dy \int_0^{L_x} dx\ \frac{\partial \psi}{\partial z}\frac{\partial}{\partial t}\left(\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) \end{aligned}$$

Here the first term is

$$
\begin{aligned}
-\int_0^{L_y} dy \int_0^{L_x} dx \left[\psi \frac{\partial}{\partial t}\left(\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right]_0^\infty &= -\int_0^{L_y} dy \int_0^{L_x} dx\, \overline{\rho}\frac{f_0^2}{N^2}\psi \frac{\partial}{\partial t}\left.\frac{\partial \psi}{\partial z}\right|_{z=0} \\
&= -\overline{\rho}\frac{f_0^2}{N^2}\int_0^{L_y} dy \int_0^{L_x} dx\, \psi\left(\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla\right)\left.\frac{\partial \psi}{\partial z}\right|_{z=0} \\
&= 0
\end{aligned}
\tag{6.218}
$$

where we have used $\overline{\rho}(z \to \infty) = 0$, (6.214) and finally also (6.207). Thus

$$
\begin{aligned}
&-\int_0^\infty dz \int_0^{L_y} dy \int_0^{L_x} dx\, \overline{\rho}\psi \frac{\partial}{\partial t}\left[\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right] \\
&= \int_0^\infty dz \int_0^{L_y} dy \int_0^{L_x} dx\, \overline{\rho}\frac{f_0^2}{N^2}\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial \psi}{\partial z}\right)^2\right]
\end{aligned}
\tag{6.219}
$$

In summary, one obtains from (6.215) the conservation law

$$
\boxed{0 = \frac{dE}{dt} \qquad E = K + A}
\tag{6.220}
$$

with

$$
\begin{aligned}
K &= \int_0^\infty dz \int_0^{L_y} dy \int_0^{L_x} dx\, \overline{\rho}\frac{1}{2}\left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2\right] \\
&= \int_0^\infty dz \int_0^{L_y} dy \int_0^{L_x} dx\, \overline{\rho}\frac{1}{2}\left[u_g^2 + v_g^2\right]
\end{aligned}
\tag{6.221}
$$

$$
\begin{aligned}
A &= \int_0^\infty dz \int_0^{L_y} dy \int_0^{L_x} dx\, \overline{\rho}\frac{f_0^2}{N^2}\frac{1}{2}\left(\frac{\partial \psi}{\partial z}\right)^2 \\
&= \int_0^\infty dz \int_0^{L_y} dy \int_0^{L_x} dx\, \overline{\rho}\frac{g^2}{N^2}\frac{1}{2}\left(\frac{\theta'}{\overline{\theta}}\right)^2
\end{aligned}
\tag{6.222}
$$

Here the conserved total energy E consists of the *kinetic energy* K and the *available potential energy* A. The latter rests in the potential-temperature fluctuations of the flow. Since, due to (6.112),

$$K = \int_0^\infty dz \int_0^{L_y} dy \int_0^{L_x} dx\, \overline{\rho} \frac{U^2 L^2}{L^2} \frac{1}{2} \left[\left(\frac{\partial \psi}{\partial \hat{x}} \right)^2 + \left(\frac{\partial \psi}{\partial \hat{y}} \right)^2 \right] \tag{6.223}$$

and, due to (6.118)

$$A = \int_0^\infty dz \int_0^{L_y} dy \int_0^{L_x} dx\, \overline{\rho} \frac{H^2}{L_{di}^2} \frac{1}{2} \left(\frac{\partial \psi}{\partial z} \right)^2 = \int_0^\infty dz \int_0^{L_y} dy \int_0^{L_x} dx\, \overline{\rho} \frac{U^2 L^2}{L_{di}^2} \frac{1}{2} \left(\frac{\partial \psi}{\partial \hat{z}} \right)^2 \tag{6.224}$$

the energy densities of structures with horizontal length scale L are in the ratio

$$\frac{K}{A} = \mathcal{O}\left(\frac{L_{di}^2}{L^2} \right) = \mathcal{O}(1) \tag{6.225}$$

Large-scale structures thus have more available potential energy than kinetic energy.

The Exchange Rate Between Available Potential Energy and Kinetic Energy

It is essential to understand how kinetic and available potential energy are transformed into each other, while total energy is conserved. For the determination of the corresponding exchange rate one needs on the one hand the quasigeostrophic vorticity equation which can be obtained by redimensionalization from (6.75):

$$\boxed{\frac{D_g}{Dt} \left(\nabla_h^2 \psi + f \right) = \frac{f_0}{\overline{\rho}} \frac{\partial}{\partial z} (\overline{\rho} w)} \tag{6.226}$$

On the other hand we need the adiabatic variant of the quasigeostrophic entropy equation, obtained by redimensionalization from (6.100) with $\hat{Q} = 0$:

$$\boxed{\frac{D_g}{Dt} \left(f_0 \frac{\partial \psi}{\partial z} \right) + w N^2 = 0} \tag{6.227}$$

Multiplying (6.226) by $-\psi \overline{\rho}$, and (6.227) by $\overline{\rho}(f_0/N^2)\partial\psi/\partial z$, and integrating both over the volume of the β-channel yields

$$\frac{dK}{dt} = \int_V dV\, \overline{\rho} f_0 w \frac{\partial \psi}{\partial z} \tag{6.228}$$

$$\frac{dA}{dt} = -\int_V dV\, \overline{\rho} f_0 w \frac{\partial \psi}{\partial z} \tag{6.229}$$

Since $\theta' \propto \partial\psi/\partial z$, available potential energy is transformed into kinetic energy if on average $w\theta' > 0$. Thus *kinetic energy is produced at the cost of available potential energy if either cold air sinks or warm air rises*.

6.2.2 The Two-Layer Model

The quasigeostrophic two-layer model satisfies energy conservation in a manner very similar to that of the continuously stratified atmosphere, shown above. Beyond this one can also recognize in its context the transformation of baroclinic kinetic energy into barotropic kinetic energy, a process of relevance in the late development of extratropical weather systems. Again we assume in the horizontal the boundary conditions of the β-channel. Moreover, already in the derivation of the equations of the two-layer model we have assumed that $\omega_b = \omega_t = 0$.

The Conservation Law

For a derivation of energy conservation we respectively multiply the two equations in (6.202) by the negative of the corresponding streamfunction, take the sum, and integrate the result over the total area of the β-channel, i.e., we form

$$-\sum_{i=1}^{2} \int_0^{L_y} dy \int_0^{L_x} dx\ \psi_i \left[\frac{\partial \pi_i}{\partial t} + J(\psi_i, \pi_i) \right] = 0 \tag{6.230}$$

By steps completely analogous to the ones in the previous chapter we obtain the conservation equation

$$\frac{dE}{dt} = 0 \tag{6.231}$$

for the total energy

$$E = K + A \tag{6.232}$$

consisting of the kinetic energy

$$K = \frac{1}{2} \int_0^{L_y} dy \int_0^{L_x} dx\ (\nabla_h \psi_1 \cdot \nabla_h \psi_1 + \nabla_h \psi_2 \cdot \nabla_h \psi_2) \tag{6.233}$$

with the horizontal streamfunction gradient

$$\nabla_h \psi_i = \frac{\partial \psi_i}{\partial x} \mathbf{e}_x + \frac{\partial \psi_i}{\partial y} \mathbf{e}_y \tag{6.234}$$

and of the available potential energy

$$A = \frac{1}{2}\int_0^{L_y} dy \int_0^{L_x} dx\ \frac{\kappa^2}{2}(\psi_1 - \psi_2)^2 \tag{6.235}$$

Here we have

$$\kappa = \sqrt{2F} \tag{6.236}$$

It has the dimension of a wave-number, and the corresponding wavelength is in midlatitudes of the order $2\pi/\kappa \approx 3000$km. Introducing the barotropic streamfunction

$$\psi = \frac{\psi_1 + \psi_2}{2} \tag{6.237}$$

and the baroclinic streamfunction

$$\tau = \frac{\psi_1 - \psi_2}{2} \tag{6.238}$$

so that

$$\psi_{1,2} = \psi \pm \tau \tag{6.239}$$

one finally obtains

$$\boxed{\frac{dE}{dt} = 0 \qquad E = K + A} \tag{6.240}$$

with

$$K = \int_0^{L_y} dy \int_0^{L_x} dx\ (\nabla_h\psi \cdot \nabla_h\psi + \nabla_h\tau \cdot \nabla_h\tau) \tag{6.241}$$

$$A = \int_0^{L_y} dy \int_0^{L_x} dx\ \kappa^2\tau^2 \tag{6.242}$$

The Exchange Rates

The kinetic energy has barotropic and baroclinic parts. In the following we examine the exchange between the two, and between kinetic and available potential energy. For this we somewhat rewrite the basic equations. The mean [(6.196) + (6.197)]/2 of the two vorticity equations yields

$$\frac{\partial}{\partial t}\nabla_h^2\psi + J(\psi, \nabla_h^2\psi + f) = -J(\tau, \nabla_h^2\tau) \tag{6.243}$$

while one obtains from the difference [(6.196) − (6.197)]/2

$$\frac{\partial}{\partial t}\nabla_h^2\tau + J(\tau, f) = -J(\tau, \nabla_h^2\psi) - J(\psi, \nabla_h^2\tau) + \frac{f_0}{\Delta p}\omega_m \tag{6.244}$$

Moreover, the entropy equation (6.198) can be written as

$$\frac{\partial \tau}{\partial t} + J(\psi, \tau) = \frac{\sigma \Delta p}{2 f_0} \omega_m \tag{6.245}$$

We then obtain the desired exchange rates as follows: Multiplication of (6.243) by -2ψ and integration over the area of the β-channel yields

$$\frac{d}{dt} \int_0^{L_y} dy \int_0^{L_x} dx\ \nabla_h \psi \cdot \nabla_h \psi = 2 \int_0^{L_y} dy \int_0^{L_x} dx\ \psi J(\tau, \nabla_h^2 \tau) \tag{6.246}$$

while multiplication of (6.244) by -2τ and the corresponding integration gives

$$\begin{aligned} &\frac{d}{dt} \int_0^{L_y} dy \int_0^{L_x} dx\ \nabla_h \tau \cdot \nabla_h \tau \\ &= -2 \int_0^{L_y} dy \int_0^{L_x} dx\ \psi J(\tau, \nabla_h^2 \tau) - \frac{2 f_0}{\Delta p} \int_0^{L_y} dy \int_0^{L_x} dx\ \tau \omega_m \end{aligned} \tag{6.247}$$

Here we have used

$$\int_0^{L_y} dy \int_0^{L_x} dx\ \tau J(\psi, \nabla_h^2 \tau) = - \int_0^{L_y} dy \int_0^{L_x} dx\ \psi J(\tau, \nabla_h^2 \tau) \tag{6.248}$$

which can easily be verified using the boundary conditions. Finally one obtains, integrating -2κ (6.245),

$$\frac{d}{dt} \int_0^{L_y} dy \int_0^{L_x} dx\ \kappa^2 \tau^2 = \frac{2 f_0}{\Delta p} \int_0^{L_y} dy \int_0^{L_x} dx\ \tau \omega_m \tag{6.249}$$

In total one thus finds that

$$C_{\psi\tau}^K = 2 \int_0^{L_y} dy \int_0^{L_x} dx\ \psi J(\tau, \nabla_h^2 \tau) \tag{6.250}$$

describes the transformation of baroclinic kinetic energy into barotropic kinetic energy, while

$$C_{AK} = -\frac{2 f_0}{\Delta p} \int_0^{L_y} dy \int_0^{L_x} dx\ \tau \omega_m \tag{6.251}$$

describes the transformation of available potential energy into baroclinic kinetic energy. The last process we have already met in the continuously stratified case. Since, due to (6.174), (6.180), and (6.238) the potential temperature in the intermediate layer at $p = p_m$ is

$$\theta_m' = 2 f_0 \left(\overline{\rho}\overline{\theta}\right)_{p=p_m} \tau \tag{6.252}$$

one finds that $C_{AK} > 0$ if $\theta_m' \omega_m < 0$, thus again if colds air sinks or warm air rises.

6.2.3 Summary

A fundamental property of quasigeostropic theory is that, beyond the material conservation of its potential vorticity, it also conserves its form of total energy.

- In the absence of friction, heat conduction, and heat sources or sinks the *sum of kinetic and available potential energy* is conserved. The latter is contained in potential-temperature fluctuations.
- *Available potential energy can be transformed into kinetic energy* if *warm air rises and cold air sinks*.
- In addition, the *two-layer model* demonstrates the process of *exchange between baroclinic kinetic energy and barotropic kinetic energy*.
- These theorems have been shown for β*-channel* geometry. They hold, however, also for other geometries.

6.3 Rossby Waves in the Stratified Atmosphere

Just as the linearized quasigeostrophic dynamics of the shallow-water equations yields free wave solutions, the stratified atmosphere does so too. As, e.g., in Fig. 6.2 such wave structures are always prominent in atmospheric data. These Rossby waves shall here be discussed in the two-layer-model approximation, followed by a treatment of the continuously stratified case. Effects of friction and orography are neglected. As boundary conditions we use those of the β-channel.

6.3.1 Rossby Waves in the Two-Layer Model

As easily verified, the two-layer-model equations (6.202) are satisfied by an altitude-independent zonal flow

$$\psi_1 = \psi_2 = -Uy \tag{6.253}$$

The perturbation ansatz

$$\psi_i = -Uy + \psi_i' \tag{6.254}$$

yields, neglecting all nonlinear terms in the infinitesimally small ψ_i',

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\nabla_h^2\psi_1' + F(\psi_2' - \psi_1')\right] + \beta\frac{\partial\psi_1'}{\partial x} = 0 \tag{6.255}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\nabla_h^2\psi_2' + F(\psi_1' - \psi_2')\right] + \beta\frac{\partial\psi_2'}{\partial x} = 0 \tag{6.256}$$

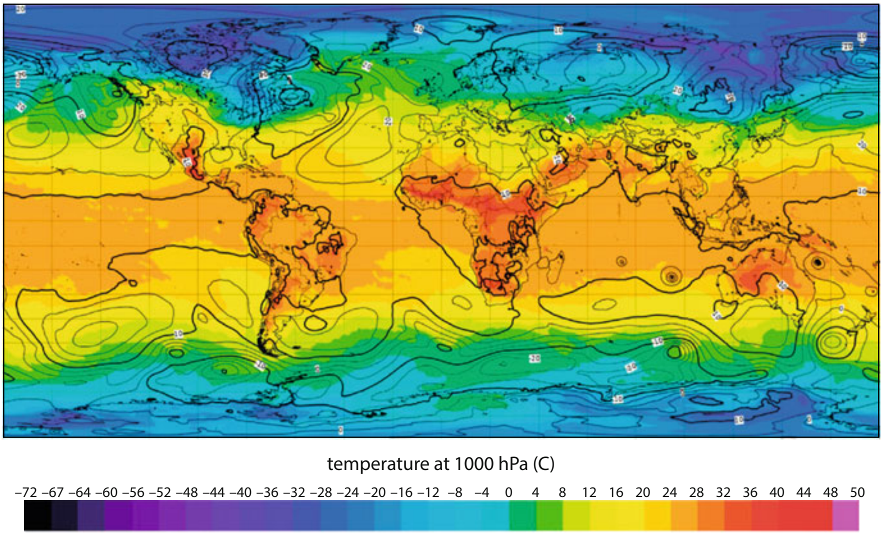

Fig. 6.2 Snapshot of geopotential (contours) and temperature (color shading) at 1000 mb. Note the clear wave structures in the geopotential. Copyright ©2021 European Center for Medium-Range Weather Forecasts (ECMWF). *Source* www.ecmwf.int. This data is published under a Creative Commons Attribution 4.0 International (CC BY 4.0). https://creativecommons.org/licenses/by/4.0/. ECMWF does not accept any liability whatsoever for any error or omission in the data, their availability, or for any loss or damage arising from their use

Again we decompose into

$$\psi' = \frac{1}{2}(\psi_1' + \psi_2') \tag{6.257}$$

$$\tau' = \frac{1}{2}(\psi_1' - \psi_2') \tag{6.258}$$

so that

$$\psi_{1,2}' = \psi' \pm \tau' \tag{6.259}$$

and form [(6.255) + (6.256)]/2, with the result

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla_h^2\psi' + \beta\frac{\partial\psi'}{\partial x} = 0 \tag{6.260}$$

This is the equation for the *barotropic mode*. The one for the *baroclinic mode* is obtained from [(6.255) − (6.256)]/2 as

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\nabla_h^2\tau' - \kappa^2\tau'\right) + \beta\frac{\partial\tau'}{\partial x} = 0 \tag{6.261}$$

Obviously in the linear limit the two modes are completely decoupled.

Since the coefficients of their respective prognostic equations do not have any spatial or temporal dependence, a solution via Fourier ansatz seems possible. As sketched in Appendix 11.5.2, both of the two streamfunctions can be expressed, due to their periodicity in x, as Fourier series,

$$\psi_i'(x, y, t) = \sum_{n=-\infty}^{\infty} \psi_i^n(y, t)\, e^{ik_n x}; \quad k_n = n\frac{2\pi}{L_x} \tag{6.262}$$

The meridional boundary conditions are

$$v_i' = \frac{\partial \psi_i'}{\partial x} = 0 \qquad \left(y = 0, L_y\right) \tag{6.263}$$

One thus has for all n

$$ik_n \psi_i^n = 0 \qquad \left(y = 0, L_y\right) \tag{6.264}$$

Hence for all $n \neq 0$

$$\psi_i^n = 0 \qquad \left(y = 0, L_y\right) \qquad (n \neq 0) \tag{6.265}$$

Following Appendix 11.5.2 one can then write

$$\psi_i^n(y, t) = \sum_{m=1}^{\infty} \psi_i^{nm}(t) \sin(l_m y) \quad l_m = m\frac{\pi}{L_y} \qquad (n \neq 0) \tag{6.266}$$

As shown in appendix F one can derive for the zonally symmetric part $n = 0$ via (6.211)

$$\psi_i^0(y, t) = D_i^0(y) + \sum_{m=1}^{\infty} \psi_i^{0m}(t) \cos(l_m y) \tag{6.267}$$

where D_i^0 is a quadratic polynomial in y. Hence

$$\begin{aligned} \psi_i'(x, y, t) = {} & D_i^0(y) \\ & + \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \psi_i^{nm}(t) \left[\delta_{n0} \cos(l_m y) + (1 - \delta_{n0}) \sin(l_m y)\right] e^{ik_n x} \end{aligned} \tag{6.268}$$

so that

$$\begin{aligned} \psi'(x, y, t) = {} & D_\psi^0(y) \\ & + \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \psi^{nm}(t) \left[\delta_{n0} \cos(l_m y) + (1 - \delta_{n0}) \sin(l_m y)\right] e^{ik_n x} \end{aligned} \tag{6.269}$$

$$\begin{aligned} \tau'(x, y, t) = {} & D_\tau^0(y) \\ & + \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \tau^{nm}(t) \left[\delta_{n0} \cos(l_m y) + (1 - \delta_{n0}) \sin(l_m y)\right] e^{ik_n x} \end{aligned} \tag{6.270}$$

where

$$\psi^{nm} = \left(\psi_1^{nm} + \psi_2^{nm}\right)/2 \tag{6.271}$$

$$\tau^{nm} = \left(\psi_1^{nm} - \psi_2^{nm}\right)/2 \tag{6.272}$$

$$D_\psi^0 = \left(D_1^0 + D_2^0\right)/2 \tag{6.273}$$

$$D_\tau^0 = \left(D_1^0 - D_2^0\right)/2 \tag{6.274}$$

Finally we express all ψ^{nm} and τ^{nm} as Fourier integrals in time so that

$$\psi'(x, y, t) = D_\psi^0(y) + \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} d\omega\, \psi^{nm\omega} \left[\delta_{n0} \cos(l_m y) + (1 - \delta_{n0}) \sin(l_m y)\right] e^{i(k_n x - \omega t)} \tag{6.275}$$

$$\tau'(x, y, t) = D_\tau^0(y) + \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} d\omega\, \tau^{nm\omega} \left[\delta_{n0} \cos(l_m y) + (1 - \delta_{n0}) \sin(l_m y)\right] e^{i(k_n x - \omega t)} \tag{6.276}$$

We first discuss the barotropic mode. (6.275) inserted into (6.260) yields

$$0 = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} d\omega\, e^{i(k_n x - \omega t)} \psi^{nm\omega} \times \left\{\delta_{n0}\omega \cos(l_m y) - (1 - \delta_{n0}) \left[(\omega - k_n U)\left(k_n^2 + l_n^2\right) + k_n \beta\right] \sin(l_m y)\right\} \tag{6.277}$$

Therefore, with $k_0 = 0$, nontrivial solutions $\psi^{nm\omega} \neq 0$ require fulfillment of the dispersion relation

$$\boxed{\omega = \omega_\psi(k_n, l_m) = k_n U - \frac{\beta k_n}{{k_n}^2 + {l_m}^2}} \tag{6.278}$$

for *barotropic Rossby waves*. Hence one has

$$\psi^{nm\omega} = \Psi^{nm} \delta\left[\omega - \omega_\psi(k_n, l_m)\right] \tag{6.279}$$

with (nearly) free complex Ψ^{nm}. Inserting these results into (6.275) leads to

$$\psi'(x, y, t) = \left[D_1^0(y) + D_2^0(y)\right]/2 + \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \Psi^{nm} e^{i[k_n x - \omega_\psi(k_n, l_m)t]} \left[\delta_{n0} \cos(l_m y) + (1 - \delta_{n0}) \sin(l_m y)\right] \tag{6.280}$$

Since ψ' is real, and $\omega_\psi(-k_n, l_m) = -\omega_\psi(k_n, l_m)$, one must have $\Psi^{nm*} = \Psi^{-nm}$ and hence with the decomposition $\Psi^{nm} = |\Psi^{nm}|e^{i\alpha_{nm}}$ into amplitude and phase, and $\Psi^{0m} \in \mathbb{R}$,

$$\psi'(x, y, t) = \left[D_1^0(y) + D_2^0(y)\right]/2 + \sum_{m=1}^{\infty} \Psi^{0m} \cos(l_m y)$$
$$+2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\Psi^{nm}| \sin(l_m y) \cos[k_n x - \omega_\psi(k_n, l_m)t + \alpha_{nm}] \quad (6.281)$$

Herein the zonally symmetric part is steady, while the longitude dependent part consists of Rossby waves, each propagating with a phase velocity

$$c_{nm} = \frac{\omega_\psi(k_n, l_m)}{k_n} = U - \frac{\beta}{k_n^2 + l_m^2} \quad (6.282)$$

in zonal direction. With respect to the basic flow they move westwards. Their spatial structure is of interest as well. At the specific time t when $\omega_\psi(k_n, l_m)t - \alpha_{nm} = \pi/2$ it is of the form $\sin(l_m y)\sin(k_n x)$, illustrated for a few examples in Fig. 6.3. One recognizes the typical sequences of pressure highs and lows characteristic for synoptic-scale weather systems in midlatitudes.

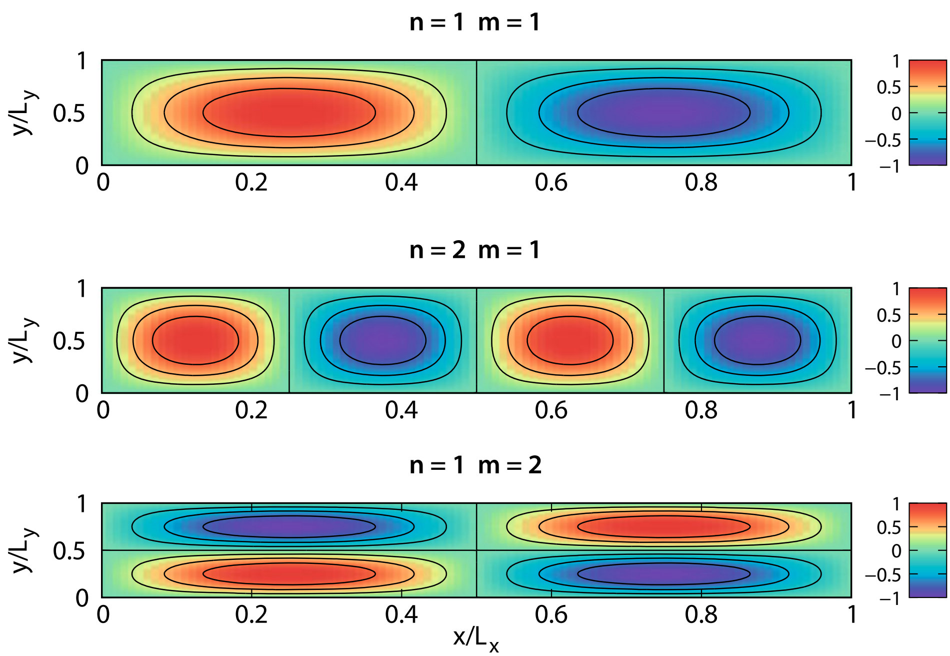

Fig. 6.3 Horizontal structure of various barotropic Rossby waves, with zonal wavenumber k_n and meridional wavenumber l_m, at time $t = (\pi/2 + \alpha_{nm})/\omega_\psi(k_n, l_m)$

The *dynamics* at the base of the westward propagation can be understood be noting that, due to the vanishing baroclinic streamfunction $\tau' = 0$ the streamfunction in both layers is identical to the barotropic streamfunction, i.e.,

$$\psi_1' = \psi_2' = \psi' \tag{6.283}$$

Hence (6.260) corresponds layer-wise to

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla_h^2\psi_i' + \beta\frac{\partial\psi_i'}{\partial x} = 0 \qquad (i = 1, 2) \tag{6.284}$$

which is each the linearization of

$$\frac{D}{Dt}(\zeta_i + f) = 0 \tag{6.285}$$

Therefore, barotropic Rossby waves conserve their absolute vorticity, which leads to westward propagation by the same mechanism as already discussed for short-wave shallow-water Rossby waves.

The calculations for the baroclinic mode are completely analogous. Use of (6.276) in (6.261) leads to the dispersion relation

$$\boxed{\omega = \omega_\tau(k_n, l_m) = k_n U - \frac{\beta k_n}{{k_n}^2 + {l_m}^2 + \kappa^2}} \tag{6.286}$$

of *baroclinic Rossby waves*. These as well have a westward directed phase velocity with respect to the basic flow. Their structure is given by

$$\begin{aligned}\tau'(x, y, t) = \left[D_1^0(y) - D_2^0(y)\right]/2 + \sum_{m=1}^{\infty} T^{0m}\cos(l_m y)\\ +2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|T^{nm}|\sin(l_m y)\cos[k_n x - \omega_\tau(k_n, l_m)t + \beta_{nm}]\end{aligned} \tag{6.287}$$

with real T^{0m} and otherwise free T^{nm} with corresponding phase β_{nm}.

Their barotropic streamfunction is $\psi' = 0$ so that the streamfunctions in the two layers are in opposite phase, i.e.,

$$\psi_1' = -\psi_2' = \tau' \tag{6.288}$$

In the case of *short baroclinic Rossby waves*, for ${K_{nm}}^2 = {k_n}^2 + {l_m}^2 >> \kappa^2$, one can neglect in (6.261) $\kappa^2\tau'$ in comparison with $\nabla_h^2\tau'$ so that approximately

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla_h^2\tau' + \beta\frac{\partial\tau'}{\partial x} = 0 \tag{6.289}$$

Hence (6.284) holds layer-wise, so that the westward propagation is caused again by the conservation of absolute vorticity.

In the case of *long baroclinic Rossby waves* with ${K_{nm}}^2 \ll \kappa^2$ the corresponding approximation is

$$-\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\kappa^2\tau' + \beta\frac{\partial \tau'}{\partial x} = 0 \tag{6.290}$$

Due to (6.288) the corresponding prognostic equations in the two layers are

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)F\left(\psi_2'-\psi_1'\right) + \beta\frac{\partial \psi_1'}{\partial x} = 0 \tag{6.291}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)F\left(\psi_1'-\psi_2'\right) + \beta\frac{\partial \psi_2'}{\partial x} = 0 \tag{6.292}$$

This is the linear approximation of

$$\frac{D}{Dt}\left[F\left(\psi_2'-\psi_1'\right)+f\right]=0 \tag{6.293}$$

$$\frac{D}{Dt}\left[F\left(\psi_1'-\psi_2'\right)+f\right]=0 \tag{6.294}$$

These waves are therefore characterized by a balance between vortex-tube stretching and planetary-vorticity advection. For an illustration we consider the situation in Fig. 6.4 where

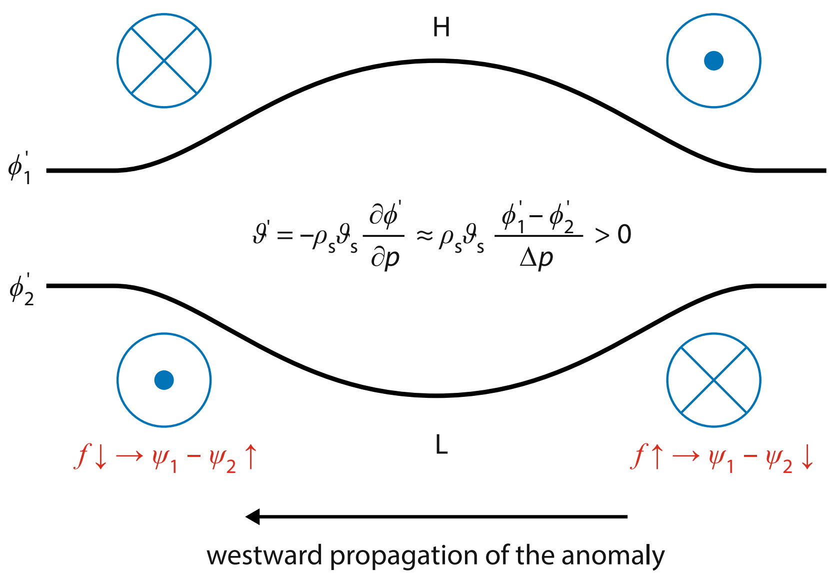

Fig. 6.4 Dynamics of a positive potential-temperature anomaly in a long-wave baroclinic Rossby wave

we see a positive anomaly of the shear streamfunction, equivalent to a positive potential-temperature anomaly. The upper-layer anomaly of geopotential and streamfunction is also positive (high-pressure anomaly), while the lower-layer anomaly is negative (low-pressure anomaly). Correspondingly the upper-layer geostrophic wind on the western flank is directed northward, while it is southward on the eastern flank. This corresponds to an increase (decrease) of planetary vorticity on the western (eastern) flank. Therefore the geopotential anomaly must increase (decrease) on the western (eastern) flank. In the lower layer conditions are opposite. The potential-temperature anomaly thus moves westwards, just as described by the dispersion relation.

6.3.2 Rossby Waves in an Isothermal Continuously Stratified Atmosphere

A continuously stratified case where Rossby waves can be treated comparatively easily is the one of a reference atmosphere with constant temperature $\overline{T}$. It has a constant scale height $H = R\overline{T}/g$ so that pressure and density have the exponential profiles

$$\overline{p}(z) = p_0 e^{-z/H} \tag{6.295}$$

$$\overline{\rho}(z) = \frac{\overline{p}}{R\overline{T}} \tag{6.296}$$

with fixed reference surface pressure p_0. Its potential temperature then is

$$\overline{\theta}(z) = \overline{T}\left(\frac{p_{00}}{p_0}\right)^{R/c_p} e^{\frac{R}{c_p}\frac{z}{H}} \tag{6.297}$$

so that

$$N^2 = \frac{g}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = \frac{R}{c_p}\frac{g}{H} \tag{6.298}$$

also is a constant.

It is easy to convince oneself that the quasigeostrophic basic equation (6.119) is satisfied in the absence of heating by the constant zonal flow

$$\psi = -Uy \tag{6.299}$$

The perturbation ansatz

$$\psi = -Uy + \psi' \tag{6.300}$$

yields, neglecting all nonlinear terms in ψ',

$$0 = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\nabla_h^2\psi' + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial\psi'}{\partial z}\right)\right] + \beta\frac{\partial\psi'}{\partial x} \tag{6.301}$$

Here the altitude dependence of the coefficients can be removed by the substitution

$$\psi' = e^{z/2H} \psi_r \tag{6.302}$$

with the result

$$0 = \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left[\nabla_h^2 \psi_r + \frac{f_0^2}{N^2} \left(\frac{\partial^2 \psi_r}{\partial z^2} - \frac{\psi_r}{4\, H^2} \right) \right] + \beta \frac{\partial \psi_r}{\partial x} \tag{6.303}$$

The rest is routine. Due to the β-channel boundary conditions

$$\begin{aligned} \psi_r\,(x, y, z, t) =& D^0(y, z) \\ &+ \sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} \int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} d\omega\; \psi^{npm\omega} \left[\delta_{n0} \cos(l_p y) + (1 - \delta_{n0}) \sin(l_p y) \right] \\ &\times e^{i(k_n x + mz - \omega t)} \end{aligned} \tag{6.304}$$

corresponding to

$$\begin{aligned} \psi'\,(x, y, z, t) =& e^{z/2H} D^0(y, z) \\ &+ \sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} \int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} d\omega\; \psi^{npm\omega} \left[\delta_{n0} \cos(l_p y) + (1 - \delta_{n0}) \sin(l_p y) \right] \\ &\times e^{z/2H + i(k_n x + mz - \omega t)} \end{aligned} \tag{6.305}$$

As dispersion relation for non-trivial $\psi^{npm\omega}$ we obtain

$$\omega = k_n U - \frac{\beta k_n}{K^2 + \dfrac{f_0^2}{N^2} m^2 + \dfrac{1}{4 L_{di}^2}} \tag{6.306}$$

where $K^2 = k_n^2 + l_p^2$ is the squared total horizontal wavenumber. Clearly, κ^2 in the two-layer model here corresponds to $m^2 f_0^2/N^2 + 1/4L_{di}^2$. The special treatment of the zonally symmetric case $n = 0$ is analogous to its treatment in the two-layer model (appendix F).

6.3.3 Summary

As in the shallow-water equations the synoptic-scale variability of the stratified atmosphere is carried by Rossby waves.

- They can be determined as solutions of the *linear* equations obtained by expanding the dynamics about a state with constant zonal flow.
- In the two-layer model one finds a barotropic and a baroclinic mode.
  - The dynamics of the *barotropic mode* corresponds to that of short-wave Rossby waves in the shallow-water equations. The advection of planetary vorticity is balanced by relative-vorticity advection, so that absolute vorticity is conserved.
  - In the *baroclinic mode* the two streamfunctions are opposite in phase so that they incorporate potential-temperature fluctuations. In the case of short wavelengths the dynamics on each layer is governed again by the conservation of absolute vorticity. At short wavelengths the planetary-vorticity advection is balanced by vortex-tube stretching.
- In the *continuously stratified atmosphere* the isothermal case can be solved analytically. Instead of just two modes, as in the two-layer case, one obtains *separate solutions for every vertical wavelength*.
- In general the β-channel boundary conditions lead to a horizontal Rossby-wave structure characterized by sequences of cyclones and anti-cyclones, as is characteristic for mid-latitude synoptic-scale weather systems.

6.4 Baroclinic Instability

The daily extratropical synoptic-scale weather is essentially carried by baroclinic waves, as also discernible in the low-level geopotential heights in Fig. 6.5. The basic mechanism in the generation of these waves is the baroclinic instability of the zonal-mean atmosphere. Differential solar heating of the atmosphere produces warm tropics and cold polar regions. The corresponding potential-temperature distribution has meridional gradients $\partial\theta/\partial y$ on the northern (southern) hemisphere which are negative (positive). Due to the thermal-wind relation this implies $\partial u/\partial z > 0$ which finds its expression in pronounced jet streams in midlatitudes (Fig. 3.8). These gradients are baroclinically unstable. The atmosphere reacts by the generation of baroclinic waves which transport heat from the tropics into the polar regions, thus working against the origin of the instability. The latter, the primary generator of synoptic weather in midlatitudes, shall be discussed here. We first consider the process in the two-layer-model approximation, followed by a discussion of the continuously stratified case. Without restriction of generality we limit ourselves to a discussion of the dynamics on the northern hemisphere.

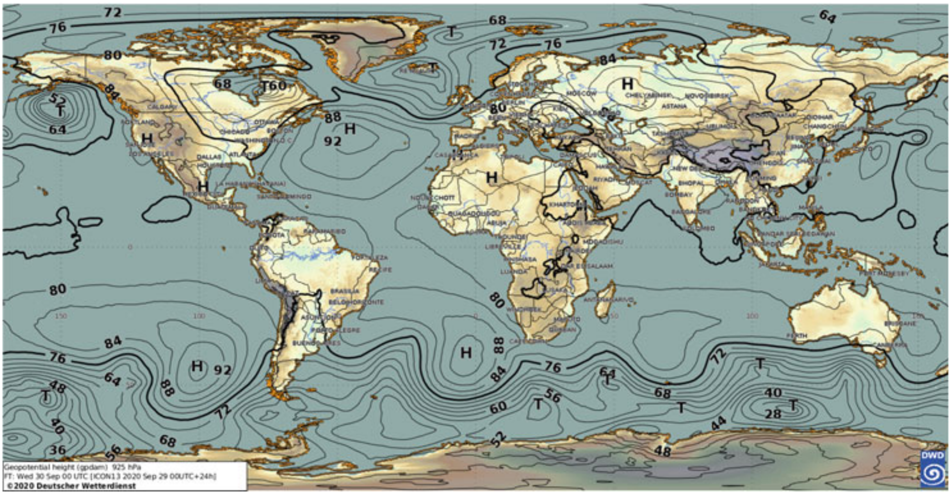

Fig. 6.5 950 mb geopotential height as predicted by DWD for September 30 in 2020. Note the chains of lows and highs discernible in the extratropics, to a large part due to baroclinic instability. *Source*: Deutscher Wetterdienst

6.4.1 Baroclinic Instability in the Two-Layer Model

The Linear Equations

As starting point we take the inviscid equations of the quasigeostrophic two-layer model in a β channel without orography. The first of these is the prognostic equation (6.243) for the barotropic streamfunction which we repeat here:

$$\frac{\partial}{\partial t}\nabla_h^2\psi + J(\psi, \nabla_h^2\psi + f) = -J(\tau, \nabla_h^2\tau) \tag{6.307}$$

The second equation we obtain by elimination of ω_m from (6.244) and (6.245):

$$\frac{\partial}{\partial t}\left(\nabla_h^2\tau - \kappa^2\tau\right) + J(\tau, f) = -J(\tau, \nabla_h^2\psi) - J(\psi, \nabla_h^2\tau - \kappa^2\tau) \tag{6.308}$$

It is easy to convince oneself that these equations are solved by

$$\psi = -Uy \tag{6.309}$$
$$\tau = -\Delta U y \tag{6.310}$$

so that

$$\psi_{1,2} = -\left(U \pm \Delta U\right) y \tag{6.311}$$
$$u_{1,2} = U \pm \Delta U \tag{6.312}$$
$$v_{1,2} = 0 \tag{6.313}$$

Here U is the barotropic part of the zonal wind velocity, and ΔU the baroclinic part. The latter corresponds to a meridional potential-temperature gradient so that potential temperature is decreasing from south to north. One also finds, e.g., by inserting into (6.245), that the solution does not entail any vertical flow:

$$\omega_m = 0 \tag{6.314}$$

We now examine the dynamics of infinitesimally small perturbations of this solution. We thus set

$$\begin{pmatrix} \psi \\ \tau \end{pmatrix} = \begin{pmatrix} -Uy \\ -\Delta U y \end{pmatrix} + \begin{pmatrix} \psi' \\ \tau' \end{pmatrix} \tag{6.315}$$

with infinitesimally small ψ' and τ'. Inserting into (6.307) and (6.308) yields, neglecting all terms nonlinear in the perturbation fields,

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla_h^2\psi' + \beta\frac{\partial\psi'}{\partial x} = -\,\Delta U\frac{\partial}{\partial x}\nabla_h^2\tau' \tag{6.316}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(\nabla_h^2\tau' - \kappa^2\tau') + \beta\frac{\partial\tau'}{\partial x} = -\,\Delta U\frac{\partial}{\partial x}\left(\nabla_h^2\psi' + \kappa^2\psi'\right) \tag{6.317}$$

In the following these equations shall be solved for arbitrary initial fields of ψ' and τ'.

The Solution of the Initial-Value Problem

As shown in Sect. 6.3.1, due to the β-channel boundary conditions the barotropic and baroclinic streamfunction can be decomposed according to (6.269) and (6.270). Inserting into (6.316) and (6.317) eliminates D_ψ^0 and D_τ^0, since they do not depend on t and x. One obtains

$$\begin{aligned} 0 = \sum_{n=-\infty}^{\infty} e^{ik_n x} \sum_{m=1}^{\infty} \Bigg\{ &\delta_{n0}\cos(l_m y) l_m^2 \frac{d\psi^{nm}}{dt} \\ &+ (1-\delta_{n0})\sin(l_m y)\Bigg[\left(\frac{d}{dt} + ik_n U\right) K_{nm}^2 \psi^{nm} \\ &-ik_n\beta\psi^{nm} + ik_n\Delta U K_{nm}^2 \tau^{nm}\Bigg]\Bigg\} \end{aligned} \tag{6.318}$$

$$\begin{aligned} 0 = \sum_{n=-\infty}^{\infty} e^{ik_n x} \sum_{m=1}^{\infty} \Bigg\{ &\delta_{n0}\cos(l_m y)\left(l_m^2 + \kappa^2\right)\frac{d\tau^{nm}}{dt} \\ &+ (1-\delta_{n0})\sin(l_m y)\Bigg[\left(\frac{d}{dt} + ik_n U\right)\left(K_{nm}^2 + \kappa^2\right)\tau^{nm} - ik_n\beta\tau^{nm} \\ &+ik_n\Delta U\left(K_{nm}^2 - \kappa^2\right)\psi^{nm}\Bigg]\Bigg\} \end{aligned} \tag{6.319}$$

yielding for the zonally symmetric parts with $n = 0$ and any m

$$\frac{d\psi^{0m}}{dt} = \frac{d\tau^{0m}}{dt} = 0 \tag{6.320}$$

i.e., the zonally symmetric part of the infinitesimally small perturbation does not develop in time. This is not the case for the longitude-dependent part, since in the subspaces for each $n \neq 0$ and m

$$\left(i\frac{\partial}{\partial t} - k_n U\right) K_{nm}^2 \psi^{nm} + k_n \beta \psi^{nm} = k_n \Delta U K_{nm}^2 \tau^{nm} \tag{6.321}$$

$$\left(i\frac{\partial}{\partial t} - k_n U\right) (K_{nm}^2 + \kappa^2)\tau^{nm} + k_n \beta \tau^{nm} = k_n \Delta U (K_{nm}^2 - \kappa^2)\psi^{nm} \tag{6.322}$$

must hold. Here again $K_{nm}^2 = k_n^2 + l_m^2$ is the squared total horizontal wave number. Now we form twice the imaginary part of ψ^{nm*} (6.321) $+ \tau^{nm*}$ (6.322). The result is[1]

$$\frac{\partial}{\partial t}\left[K_{nm}^2 |\psi^{nm}|^2 + \left(K_{nm}^2 + \kappa^2\right)|\tau^{nm}|^2\right] = 2k_n \Delta U \kappa^2 \mathrm{Im}(\psi^{nm*}\tau^{nm}) \tag{6.323}$$

Hence, in the absence of any velocity shear ΔU, the *pseudoenergy*

$$E' = K_{nm}^2 |\psi^{nm}|^2 + \left(K_{nm}^2 + \kappa^2\right)|\tau^{nm}|^2 \tag{6.324}$$

is conserved within linear dynamics. Thus motivated we define the vector

$$\boldsymbol{\Psi}^{nm}(t) = \begin{pmatrix} K_{nm}\psi^{nm} \\ \sqrt{K_{nm}^2 + \kappa^2}\tau^{nm} \end{pmatrix} \tag{6.325}$$

with pseudoenergy norm

$$|\boldsymbol{\Psi}|^2 = E' \tag{6.326}$$

The transformed Eqs. (6.321) and (6.322) can then be written in the compact form

$$\boxed{\left(i\frac{\partial}{\partial t} - k_n U\right)\boldsymbol{\Psi}^{nm} = H_{nm}\boldsymbol{\Psi}^{nm}} \tag{6.327}$$

with

$$\boxed{H_{nm} = \begin{pmatrix} \omega_\psi & \alpha \\ \alpha - \gamma & \omega_\tau \end{pmatrix}} \tag{6.328}$$

[1] The asterisk denotes complex conjugation.

Here

$$\omega_\psi = -\frac{\beta k_n}{K_{nm}^2} \tag{6.329}$$

$$\omega_\tau = -\frac{\beta k_n}{K_{nm}^2 + \kappa^2} \tag{6.330}$$

are the intrinisc frequencies of barotropic and baroclinic Rossby waves in a reference frame moving zonally at velocity U, and

$$\alpha = \frac{k_n \Delta U K_{nm}}{\sqrt{K_{nm}^2 + \kappa^2}} \tag{6.331}$$

$$\gamma = \frac{\kappa^2}{K_{nm}^2}\alpha \tag{6.332}$$

are contributions to H_{nm}, nonzero only if a zonal-wind shear $\Delta U \neq 0$ exists.

Fourier transformation of (6.327) in time, so that

$$\mathbf{\Psi}^{nm}(t) = \int_{-\infty}^{\infty} d\omega \mathbf{\Psi}^{nm\omega} e^{-i\omega t} \tag{6.333}$$

yields the eigenvalue equation

$$\hat{\omega}\mathbf{\Psi}^{nm\omega} = H_{nm}\mathbf{\Psi}^{nm\omega} \tag{6.334}$$

where

$$\hat{\omega} = \omega - k_n U \tag{6.335}$$

is the intrinsic frequency observed in a reference frame moving at velocity U in zonal direction. Non-trivial solutions $\mathbf{\Psi}^{nm\omega}$ must be eigenvectors of H_{nm}. The two eigenvalues $\hat{\omega}_{1,2}$ are determined via

$$\det\left(H_{nm} - \hat{\omega}_i I\right) = 0 \tag{6.336}$$

They hence solve

$$\boxed{(\hat{\omega}_i - \omega_\psi)(\hat{\omega}_i - \omega_\tau) = \alpha\,(\alpha - \gamma)} \tag{6.337}$$

Since the coefficients of H_{nm} are all real, the two eigenvalues are either real or a complex-conjugate pair, i.e., $\hat{\omega}_1 = \hat{\omega}_2^*$. Up to a normalization factor the corresponding eigenvectors $\mathbf{\Psi}_{1,2}^{nm}$ are determined by

$$\boxed{\hat{\omega}_i \mathbf{\Psi}_i^{nm} = H_{nm}\mathbf{\Psi}_i^{nm}} \tag{6.338}$$

The general solution of (6.327) is therefore

$$\mathbf{\Psi}^{nm}(t) = \sum_{j=1}^{2} \mathbf{\Psi}_j^{nm} A_j^{nm} e^{-i\omega_j t} \tag{6.339}$$

or

$$\boldsymbol{\Psi}^{nm}(t) = \sum_{j=1}^{2} \boldsymbol{\Psi}_j^{nm} A_j^{nm} e^{-i(k_n U + \hat{\omega}_j)t} \tag{6.340}$$

For the determination of the A_j^{nm} from the initial state we additionally consider the adjoint problem

$$\hat{\alpha}_i \boldsymbol{\Phi}_i^{nm} = H_{nm}^t \boldsymbol{\Phi}_i^{nm} \tag{6.341}$$

As one can easily convince oneself, the eigenvalues are the same as above. Because they are either real-valued or a complex-conjugate pair, we can order them as

$$\hat{\alpha}_i = \hat{\omega}_i^* \tag{6.342}$$

Since

$$\hat{\omega}_j \left(\boldsymbol{\Phi}_i^{nm}\right)^\dagger \boldsymbol{\Psi}_j^{nm} = \left(\boldsymbol{\Phi}_i^{nm}\right)^\dagger H_{nm} \boldsymbol{\Psi}_j^{nm} = \left(H_{nm}^t \boldsymbol{\Phi}_i^{nm}\right)^\dagger \boldsymbol{\Psi}_j^{nm} = \hat{\omega}_i \left(\boldsymbol{\Phi}_i^{nm}\right)^\dagger \boldsymbol{\Psi}_j^{nm} \tag{6.343}$$

one has

$$\left(\hat{\omega}_j - \hat{\omega}_i\right) \left(\boldsymbol{\Phi}_i^{nm}\right)^\dagger \boldsymbol{\Psi}_j^{nm} = 0 \tag{6.344}$$

and therefore for $i \neq j$

$$\left(\boldsymbol{\Phi}_i^{nm}\right)^\dagger \boldsymbol{\Psi}_j^{nm} = 0 \tag{6.345}$$

Eigenvectors to different eigenvalues are orthogonal to each other. Without loss of generality we choose their normalization factors so that

$$\left(\boldsymbol{\Phi}_i^{nm}\right)^\dagger \boldsymbol{\Psi}_j^{nm} = \delta_{ij} \tag{6.346}$$

The initial perturbation

$$\boldsymbol{\Psi}^{nm}(0) = \sum_{j=1}^{2} \boldsymbol{\Psi}_j^{nm} A_j^{nm} \tag{6.347}$$

can be projected directly onto the eigenvectors, with the result

$$A_j^{nm} = \left(\boldsymbol{\Phi}_j^{nm}\right)^\dagger \boldsymbol{\Psi}^{nm}(0) \tag{6.348}$$

The solution in physical space is reconstructed by determining, according to (6.325), for the j-th eigenvector ψ_j^{nm} and τ_j^{nm} so that

$$\boldsymbol{\Psi}_j^{nm} = \begin{pmatrix} K_{nm} \psi_j^{nm} \\ \sqrt{K_{nm}^2 + \kappa^2} \tau_j^{nm} \end{pmatrix} \tag{6.349}$$

where the index j does *not* indicate a layer! Then the general solution is

$$\begin{aligned}\begin{pmatrix}\psi'\\ \tau'\end{pmatrix}(x,y,t) = &\begin{pmatrix}D_\psi^0\\ D_\tau^0\end{pmatrix}(y)\\ &+\Re \sum_{n=-\infty}^{\infty} e^{ik_n x} \sum_{m=1}^{\infty}\left[\delta_{n0}\begin{pmatrix}\psi^{0m}\\ \tau^{0m}\end{pmatrix}\cos(l_m y)\right.\\ &\left.+(1-\delta_{n0})\sin(l_m y)\sum_{j=1}^{2} A_j^{nm}\begin{pmatrix}\psi_j^{nm}\\ \tau_j^{nm}\end{pmatrix} e^{-i(k_n U+\hat{\omega}_j)t}\right]\end{aligned} \tag{6.350}$$

where we also express the fact that only real fields can result from real initial conditions. This solution is brought into a more explicit form in appendix G.

Baroclinic Waves and Their Structure

In the following we consider three cases. For transparency of notation we often suppress the indices n and m.

No Zonal-Wind Shear ($\Delta U = 0$) In the absence of zonal-wind shear one has $\alpha(\alpha - \gamma) = 0$. With this one obtains as solutions a superposition of the well-known *free Rossby waves* with

$$\hat{\omega}_{1,2} = \omega_{\psi,\tau} \tag{6.351}$$

Zonal-Wind Shear, but no β-effect ($\Delta U \neq 0$, $\beta = 0$) In this case we have

$$\omega_\psi = \omega_\tau = 0 \tag{6.352}$$

The intrinsic frequencies thus satisfy

$$\hat{\omega}_i^2 = \alpha(\alpha - \gamma) \tag{6.353}$$

or

$$\hat{\omega}_i^2 = k^2 \Delta U^2 \frac{K_{nm}^2 - \kappa^2}{K_{nm}^2 + \kappa^2} \tag{6.354}$$

The interesting case is the one of *long* waves with $K_{nm}^2 < \kappa^2$. For these we get

$$\hat{\omega}_{1,2} = \pm i\Gamma \tag{6.355}$$

with a *growth rate*

$$\Gamma = k\Delta U \sqrt{\frac{\kappa^2 - K_{nm}^2}{\kappa^2 + K_{nm}^2}} \tag{6.356}$$

The time dependence of the perturbation is

$$\begin{aligned}
&\sum_{j=1}^{2} A_j^{nm} \begin{pmatrix} \psi_j^{nm} \\ \tau_j^{nm} \end{pmatrix} e^{-i(k_n U + \hat{\omega}_j)t} \\
&= A_1^{nm} \begin{pmatrix} \psi_1^{nm} \\ \tau_1^{nm} \end{pmatrix} e^{-ik_n Ut} e^{\Gamma t} + A_2^{nm} \begin{pmatrix} \psi_2^{nm} \\ \tau_2^{nm} \end{pmatrix} e^{-ik_n Ut} e^{-\Gamma t}
\end{aligned} \tag{6.357}$$

The first part thus grows exponentially! This is the *baroclinic instability*.[2] From a virtually arbitrary initial perturbation, with a nonzero projection onto such a baroclinic wave, the latter will grow and progressively dominate. This growth can only be halted by nonlinear dynamics. Since an initial perturbation typically has contributions from different n and m, one usually observes a superposition of growing waves, which again will finally be dominated by the fastest growing mode. We note the following:

- Without β-effect one finds an instability for all ΔU.
- This necessitates $K < \kappa$. The corresponding wavelength is in midlatitudes

$$\lambda = \frac{2\pi}{K} > \frac{2\pi}{\kappa} \approx 3000\,\text{km} \tag{6.358}$$

  Baroclinic instability is characterized by relatively large wavelengths. The extension of the corresponding pressure anomalies is approximately $\lambda/2$. This is consistent with the synoptic-scale estimate $L = 1000\,\text{km}$.
- The growth rate is largest at

$$\frac{\partial\Gamma}{\partial k} = \frac{\partial\Gamma}{\partial l} = 0 \tag{6.359}$$

  This leads to $l = 0$ and $K_{nm}^2 = \left(\sqrt{2} - 1\right)\kappa^2$, and thus

$$(k, l)_{\max} = \left(\sqrt{\sqrt{2} - 1}\kappa, 0\right) \tag{6.360}$$

  However, since the smallest possible meridional wavenumber component is $l = \pi/L_y$, the more precise result is

$$(k, l)_{\max} = \left\{\kappa \sqrt{\left[\sqrt{2\left(1 + \frac{\pi^2}{\kappa^2 L_y^2}\right)} - 1\right] - \frac{\pi^2}{\kappa^2 L_y^2}}, \frac{\pi}{L_y}\right\} \tag{6.361}$$

  For $L_y \gg \pi/\kappa \approx 1500\,\text{km}$, however, the difference is negligible. The zonal wavelength for maximum growth then is in midlatitudes

[2] A critical reader might remark that complex frequencies are not really admitted by Fourier transforms. The mathematically cleanest way would be applying Laplace transforms, but the final results are basically the same.

$$\lambda = \frac{2\pi}{k_{\max}} \approx \frac{2\pi}{\sqrt{\sqrt{2}-1}\kappa} \approx 5000\,\text{km} \tag{6.362}$$

For a determination of the *structure* of a baroclinically unstable wave we Fourier transform (6.321) in time, obtaining

$$\hat{\omega} K_{nm}^2 \psi^{nm\omega} + k_n \beta \psi^{nm\omega} = k_n \Delta U K_{nm}^2 \tau^{nm\omega} \tag{6.363}$$

With $\beta = 0$, (6.355), and (6.356) this leads for the growing baroclinic wave to

$$\tau_1^{nm} = i \frac{\Gamma}{k_n \Delta U} \psi_1^{nm} = \sqrt{\frac{\kappa^2 - K_{nm}^2}{\kappa^2 + K_{nm}^2}} e^{i\frac{\pi}{2}} \psi_1^{nm} \tag{6.364}$$

Thus the corresponding streamfunctions in the two layers are

$$\begin{aligned}(\psi_{1,2}^{nm})_1 &= \psi_1^{nm} \pm \tau_1^{nm} = \psi_1^{nm}\left(1 \pm i\sqrt{\frac{\kappa^2 - K_{nm}^2}{\kappa^2 + K_{nm}^2}}\right) \\ &= \frac{\sqrt{2}\kappa}{\sqrt{\kappa^2 + K_{nm}^2}} e^{\pm i\epsilon} \psi_1^{nm}\end{aligned} \tag{6.365}$$

where

$$\epsilon = \arctan\sqrt{\frac{\kappa^2 - K_{nm}^2}{\kappa^2 + K_{nm}^2}} \tag{6.366}$$

is half the phase difference between the upper and the lower layer. In the subspace of the wavenumber combination with largest growth we may now decompose $A_1^{nm}\psi_1^{nm} = A_\psi e^{i\alpha}$, so that the barotropic streamfunction of the growing part is, due to (6.350),

$$\begin{aligned}\psi'(x, y, t) &= \Re\left[A_1^{nm}\psi_1^{nm} \sin(l_m y)\, e^{i(k_n x - k_n U t)} e^{\Gamma t}\right] \\ &= A_\psi \sin(l_m y) \cos\left(k_n x - k_n U t + \alpha\right) e^{\Gamma t}\end{aligned} \tag{6.367}$$

and thus the baroclinic streamfunction, via (6.364),

$$\begin{aligned}\tau'(x, y, t) &= \Re\left[A_1^{nm}\tau_1^{nm} \sin(l_m y)\, e^{i(k_n x - k_n U t)} e^{\Gamma t}\right] \\ &= \sqrt{\frac{\kappa^2 - K_{nm}^2}{\kappa^2 + K_{nm}^2}} A_\psi \sin(l_m y) \cos(k_n x - k_n U t + \alpha + \frac{\pi}{2}) e^{\Gamma t}\end{aligned} \tag{6.368}$$

It thus leads the barotropic streamfunction by a phase difference $\pi/2$, correspondingly in x by $\Delta x = \lambda/4$, where $\lambda = 2\pi/k_n$ is the zonal wavelength of the wave. The streamfunctions in the two layers thus are

$$\psi'_{1,2}(x, y, t) = \psi \pm \tau$$
$$= \frac{\sqrt{2}\kappa}{\sqrt{\kappa^2 + K_{nm}^2}} A_\psi \sin(l_m y)\cos(k_n x - k_n U t + \alpha \pm \epsilon)e^{\Gamma t} \quad (6.369)$$

The upper-layer streamfunction leads the barotropic streamfunction in phase by ϵ, or in x by $\Delta x = \lambda\epsilon/2\pi$. By the same difference the lower-layer streamfunction follows the barotropic streamfunction. The resulting *westward tilt* of the phase with increasing altitude is sketched in Fig. 6.6. Note that the middle-layer potential temperature θ'_m is given, due to (6.252), up to a constant factor by the baroclinic streamfunction. The meridional-wind velocity there is $v' = \partial\psi'/\partial x$, leading ψ' in phase by $\pi/2$. Thus v' is in phase with θ'. The westward tilt thus implies that *warm air is transported northwards, and cold air southwards*. The baroclinic wave thus operates against the cause of its instability.

Zonal-Wind Shear and β-Effect ($\Delta U \neq 0, \beta \neq 0$) In the general case the solution of (6.337) is

$$\hat{\omega}_{1,2} = \frac{\omega_\psi + \omega_\tau}{2} \pm \sqrt{\left(\frac{\omega_\psi - \omega_\tau}{2}\right)^2 + \alpha(\alpha - \gamma)} \quad (6.370)$$

The argument of the square root must be negative for an instability. This implies

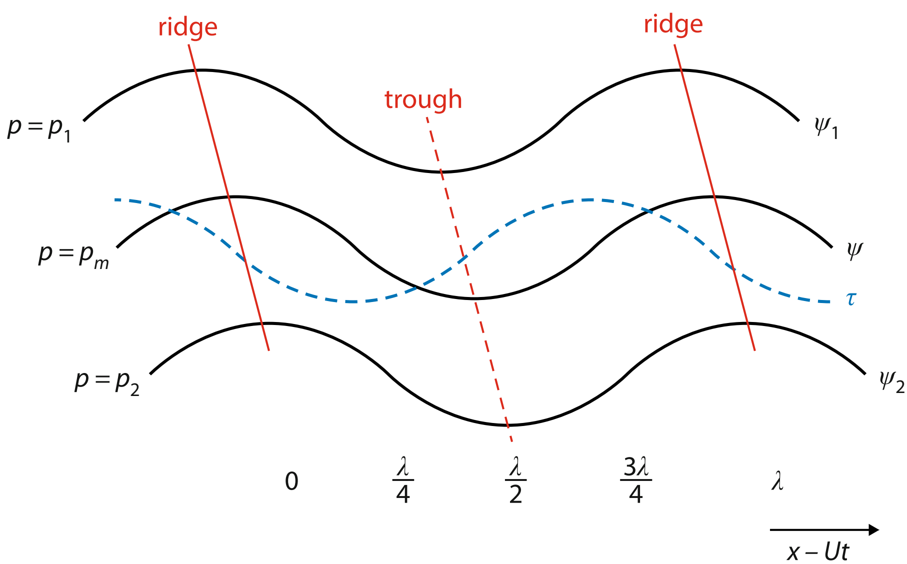

Fig. 6.6 Longitude–altitude structure of a growing baroclinic wave. Note the westward tilt of the phase. The potential temperature in the intermediate layer, proportional to τ, is in phase with the meridional wind $v = \partial\psi/\partial x$ at the same altitude

$$\alpha\,(\alpha-\gamma)<0 \quad \text{and} \quad \left(\frac{\omega_\psi-\omega_\tau}{2}\right)^2<\alpha\,(\gamma-\alpha)$$

leading to

$$\boxed{K_{nm}^2<\kappa^2 \qquad \text{and} \qquad \beta^2\kappa^4<4\Delta U^2K_{nm}^4(\kappa^4-K_{nm}^4)} \tag{6.371}$$

The β-effect thus stabilizes the flow. At a given total K_{nm} an instability is only possible if

$$\Delta U^2>G\,(K_{nm})=\frac{\beta^2\kappa^4}{4K_{nm}^4(\kappa^4-K_{nm}^4)} \tag{6.372}$$

This is also sketched in Fig. 6.7. No instability is possible if ΔU^2 is below the minimum of G. The latter is

$$\min G\,(K)=\frac{\beta^2}{\kappa^4} \quad \text{at} \quad K_{nm}^2=\frac{\kappa^2}{\sqrt{2}}$$

Thus the flow is only unstable if

$$\boxed{\Delta U^2>\frac{\beta^2}{\kappa^4}} \tag{6.373}$$

The following is also important:

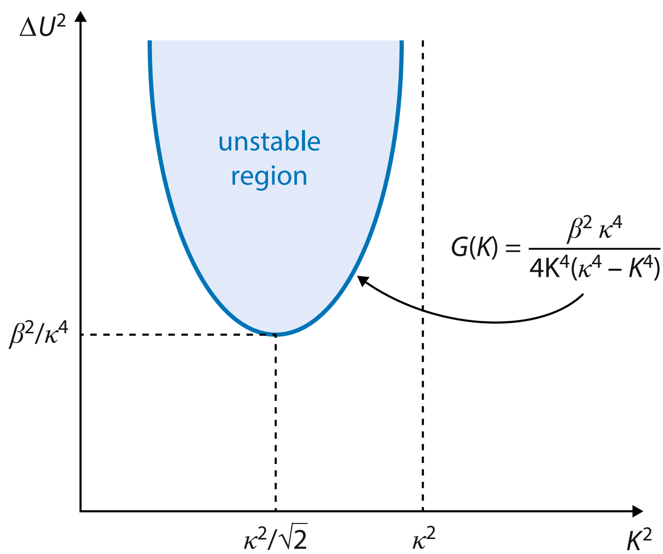

Fig. 6.7 For a quasigeostrophic two-layer model, the baroclinically unstable region in a $K^2-\Delta U^2$-diagram

- In the unstable regime the growth rate

$$\Gamma = \sqrt{\alpha(\gamma - \alpha) - \left(\frac{\omega_\psi - \omega_\tau}{2}\right)^2} \tag{6.374}$$

is again largest at $l = \pi/L_y$. The zonal wavenumber with largest instability is close to

$$k_{\max} \approx \frac{\kappa}{2^{1/4}} \tag{6.375}$$

corresponding to a wavelength $\lambda = 2^{1/4} 2\pi/\kappa \approx 3000\,\mathrm{km}$.

- The *latitude dependence* of the potential for baroclinic instability is also interesting. The definitions of κ and β imply

$$\frac{\beta}{\kappa^2} = \frac{\sigma \Delta p^2}{4\Omega a} \frac{\cos\phi_0}{\sin^2\phi_0} \tag{6.376}$$

The minimum zonal-wind shear which must be exceeded for an instability also depends on the reference latitude ϕ_0. On the synoptic scale, which here is the focus, the tropics are much more stable than the midlatitudes (Fig. 6.8). At $\phi_0 = 45°$ one finds that $\Delta U > 3\,\mathrm{m/s}$ must be satisfied, corresponding to a zonal-wind shear between the two layers of 6 m/s.

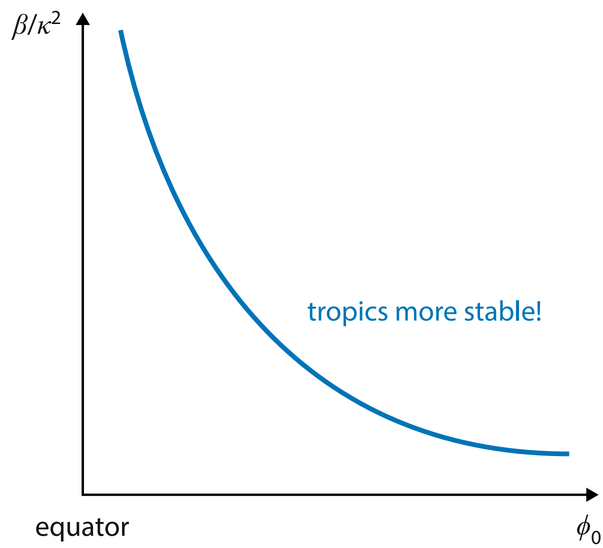

Fig. 6.8 Latitude dependence of the minimum zonal-wind shear necessary for a baroclinic instability. The tropics are much more stable than the midlatitudes

Mechanisms and Energetics

For the further analysis of the mechanisms and energetics of the baroclinic instability we linearize (6.243), (6.244), and (6.245) directly about

$$\begin{pmatrix} \psi \\ \tau \\ \omega_m \end{pmatrix} = \begin{pmatrix} -Uy \\ -\Delta U y \\ 0 \end{pmatrix} \tag{6.377}$$

The result is

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla_h^2\psi' + \beta\frac{\partial\psi'}{\partial x} = -\Delta U\frac{\partial}{\partial x}\nabla_h^2\tau' \tag{6.378}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla_h^2\tau' + \beta\frac{\partial\tau'}{\partial x} = -\Delta U\frac{\partial}{\partial x}\nabla_h^2\psi' + \frac{f_0}{\Delta p}\omega_m' \tag{6.379}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\tau' - \Delta U\frac{\partial\psi'}{\partial x} = \frac{\sigma\Delta p}{2f_0}\omega_m' \tag{6.380}$$

Together with the boundary conditions of the β-channel the integral $-2\int d^2x\left[\psi'\right.$ $(6.378) + \tau'\,(6.379)\left.\right]$ is

$$\frac{dK'}{dt} = -\frac{2f_0}{\Delta p}\int_0^{L_y} dy\int_0^{Lx} dx\;\; \tau'\omega_m' \tag{6.381}$$

$$K' = \int_0^{L_y} dy\int_0^{Lx} dx\;\left(\nabla_h\psi'\cdot\nabla_h\psi' + \nabla_h\tau'\cdot\nabla_h\tau'\right) \tag{6.382}$$

while $2\kappa^2\int d^2x\tau'$ (6.380) yields

$$\frac{dA'}{dt} = \frac{2f_0}{\Delta p}\int_0^{L_y} dy\int_0^{Lx} dx\;\; \tau'\omega_m' + 2\Delta U\kappa^2\int_0^{L_y} dy\int_0^{Lx} dx\;\; \tau'\frac{\partial\psi'}{\partial x} \tag{6.383}$$

$$A' = \int_0^{L_y} dy\int_0^{Lx} dx\;\kappa^2\tau'^2 \tag{6.384}$$

A perturbation grows if

$$\frac{d}{dt}(K + A) > 0 \tag{6.385}$$

which is equivalent to

$$2\Delta U\kappa^2 \int_0^{L_y} dy \int_0^{Lx} dx \;\; \tau' \frac{\partial \psi'}{\partial x} > 0 \tag{6.386}$$

In the integral one must also have predominantly $\tau'\partial\psi'/\partial x > 0$, which is equivalent to $f_0\theta'_m v' > 0$ so that warm air is transported to the pole and cold air to the equator. As a consequence, available potential energy $\bar{A}$ of the basic flow is reduced and exchanged into available potential energy A' of the perturbations. As we have already convinced us above, this is the case for growing baroclinic waves. Vice versa an oppositely directed mean transport so that $f_0\theta'_m v' < 0$ implies that the perturbation is damped.

The other partial process we have already met in the discussion of the general energetics of the two-layer model: If

$$-\frac{2f_0}{\Delta p} \int_0^{L_y} dy \int_0^{Lx} dx \;\; \tau'\omega'_m > 0 \tag{6.387}$$

available potential energy A' of the perturbation is transformed into kinetic energy K' of the perturbation. Since up to a factor τ' is equivalent to θ'_m, the condition for this is that warm air rises and cold air sinks. Also this can be checked directly for growing baroclinic waves. For simplicity we limit ourselves for this to the case $\beta = 0$: First, the Fourier transform of (6.380) yields

$$\omega_m^{nm\omega} = -\frac{2if_0}{\sigma\Delta p}(\hat{\omega}\tau^{nm\omega} + k_n\Delta U\psi^{nm\omega}) \tag{6.388}$$

Together with $\hat{\omega} = \hat{\omega}_1 = i\Gamma$, (6.356), and (6.364) one obtains from this for the growing baroclinic wave

$$(\omega_m)_1^{nm} = -\frac{4f_0}{\sigma\Delta p}\Gamma\frac{K_{nm}^2}{\kappa^2 - K_{nm}^2}\tau_1^{nm} \tag{6.389}$$

One thus sees that in this wave ω'_m and τ' have opposite phases, as we could show. Figures 6.9 and 6.10 summarize what we have learned.

6.4.2 Baroclinic Instability in a Continuously Stratified Atmosphere

The Linear Equations

Starting point for the analysis of a continuously stratified atmosphere is the quasigeostrophic potential-vorticity conservation Eq. (6.119) without heating. As boundary conditions we use in the horizontal those of the β-channel, i.e., (6.204–6.206). Orography and heating are neglected, so that the vertical boundary condition at the ground is given by (6.207). As upper boundary condition we can use (6.208). In the case of an approximation where a solid upper boundary of the atmosphere is assumed at altitude H, we can use alternatively, in analogy with (6.207),

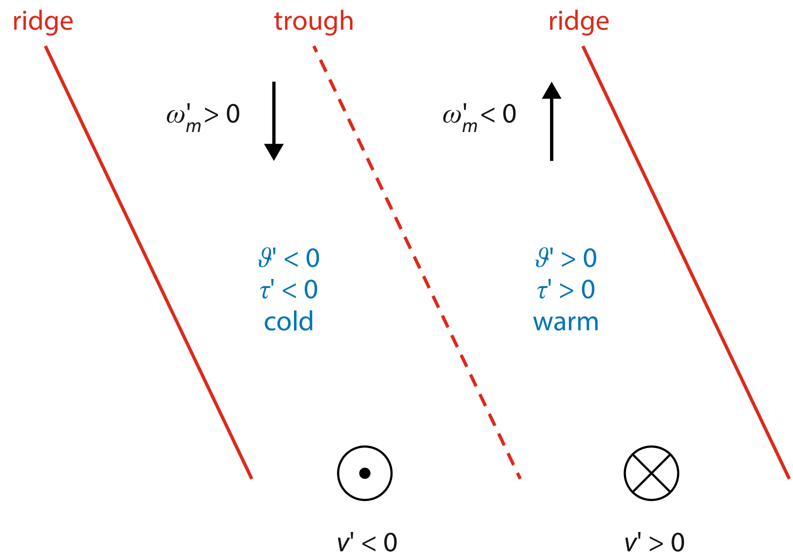

Fig. 6.9 Longitude–altitude section of the exchange processes determining the energetics of a growing baroclinic wave in the northern hemisphere: The meridional heat transport conveys warm (*cold*) air to the north (*south*). The vertical transport leads to upward (*downward*) motion of warm (*cold*) air masses

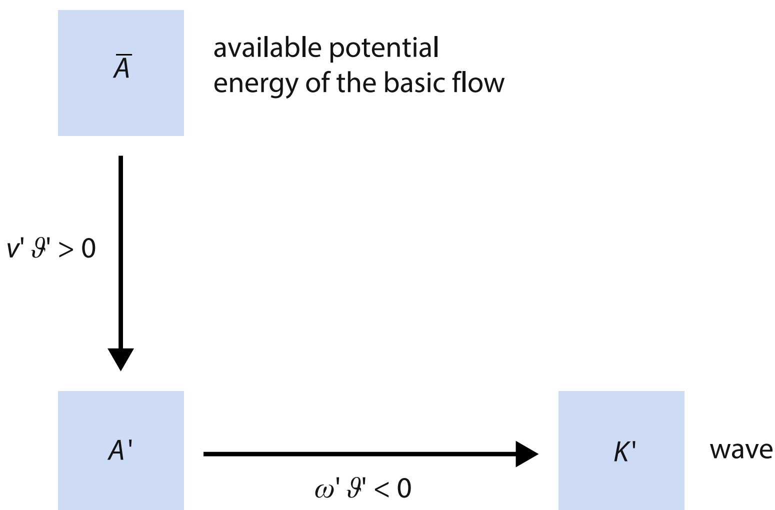

Fig. 6.10 Schematic representation of the energy-exchange processes in a baroclinic instability. The meridional transport transforms available potential energy in the basic flow into available potential energy of the growing wave, while vertical heat transport transforms the latter into kinetic energy of the wave

$$\left.\frac{D_g}{Dt}\frac{\partial \psi}{\partial z}\right|_{z=H} = 0 \tag{6.390}$$

It is easy to convince oneself that these equations are solved by a zonally symmetric and steady streamfunction

$$\psi = \bar{\psi}(y, z) \tag{6.391}$$

with corresponding horizontal wind fields

$$\bar{u} = -\frac{\partial \bar{\psi}}{\partial y} \tag{6.392}$$

$$\bar{v} = 0 \tag{6.393}$$

At a sufficiently strong vertical gradient of $\bar{u}$ we again expect a baroclinic instability.

With the aim of a corresponding analysis we linearize the equations about this basic flow, i.e., we use the perturbation ansatz

$$\psi = \bar{\psi}(y, z) + \psi'(x, y, z, t) \tag{6.394}$$

with infinitesimally small ψ', insert this into the equations and then neglect all contributions which are nonlinear in ψ'. Inserting this into (6.119) thus leads to

$$\boxed{\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\pi' + \frac{\partial \psi'}{\partial x}\frac{\partial \bar{\pi}}{\partial y} = 0} \tag{6.395}$$

where

$$\pi' = \nabla_h^2 \psi' + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\frac{\overline{\rho} f_0^2}{N^2}\frac{\partial \psi'}{\partial z}\right) \tag{6.396}$$

is the quasigeostrophic potential vorticity of the perturbation, and

$$\bar{\pi} = \frac{\partial^2 \bar{\psi}}{\partial y^2} + f + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\frac{\overline{\rho} f_0^2}{N^2}\frac{\partial \bar{\psi}}{\partial z}\right) \tag{6.397}$$

that of the basic flow, with meridional gradient

$$\frac{\partial \bar{\pi}}{\partial y} = -\frac{\partial^2 \bar{u}}{\partial y^2} + \beta - \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\frac{\overline{\rho} f_0^2}{N^2}\frac{\partial \bar{u}}{\partial z}\right) \tag{6.398}$$

The linearization of the meridional boundary conditions (6.206) leads to

$$\boxed{\frac{\partial \psi'}{\partial x} = 0 \qquad (y = 0, L_y)} \tag{6.399}$$

while the vertical boundary condition (6.207) yields

$$0 = \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\frac{\partial \psi'}{\partial z} + \frac{\partial \psi'}{\partial x}\frac{\partial}{\partial y}\frac{\partial \bar{\psi}}{\partial z} \qquad (z = 0) \tag{6.400}$$

or

$$0 = \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\frac{\partial \psi'}{\partial z} - \frac{\partial \psi'}{\partial x}\frac{\partial \bar{u}}{\partial z} \qquad (z = 0) \tag{6.401}$$

If needed such a boundary condition can also be applied at a solid upper boundary at $z = H$.

As in the case of the two-layer model we can assume, without restriction of generality, that the perturbation can be represented as

$$\psi' = \sum_k \int_{-\infty}^{\infty} d\omega e^{i(kx-\omega t)} \hat{\psi}(k, y, z, \omega) \tag{6.402}$$

where the contributing zonal wavenumbers are

$$k = n\frac{2\pi}{L_x} \qquad (n \in \mathbb{Z}) \tag{6.403}$$

Inserting this into (6.395) yields

$$(\omega - k\bar{u})\left[-k^2\hat{\psi} + \frac{\partial^2 \hat{\psi}}{\partial y^2} + \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left(\bar{\rho}\frac{f_0^2}{N^2}\frac{\partial \hat{\psi}}{\partial z}\right)\right] - k\hat{\psi}\frac{\partial \bar{\pi}}{\partial y} = 0 \tag{6.404}$$

while the boundary conditions (6.399) and (6.401) lead to

$$\hat{\psi} = 0 \qquad \left(y = 0, L_y \quad k \neq 0\right) \tag{6.405}$$

and

$$(\omega - k\bar{u})\frac{\partial \hat{\psi}}{\partial z} + k\hat{\psi}\frac{\partial \bar{u}}{\partial z} = 0 \qquad (z = 0) \tag{6.406}$$

Similar to the two-layer model we do not expect any wave growth in the zonally symmetric case $k = 0$ so that we limit ourselves in the following to longitude dependent perturbations with $k \neq 0$.

The Rayleigh Theorem

A closed analytical treatment of the linear equations is only possible in special cases. Beyond these, however, there is a general theorem that tells us under which conditions a zonally symmetric flow can become unstable within the framework of quasigeostrophic theory at all. For this we assume à priori that

$$\omega = \omega_r + i\Gamma \qquad (\Gamma > 0) \tag{6.407}$$

and examine under which cases this does not lead to a contradiction.

We first take the real and imaginary part of (6.404)/$(\omega - k\bar{u})$:

$$\frac{\partial^2 \hat{\psi}_r}{\partial y^2} + \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left(\bar{\rho}\frac{f_0^2}{N^2}\frac{\partial \hat{\psi}_r}{\partial z}\right) - \left(k^2 + k\delta_r\frac{\partial \bar{\pi}}{\partial y}\right)\hat{\psi}_r - k\delta_i\frac{\partial \bar{\pi}}{\partial y}\hat{\psi}_i = 0 \qquad (6.408)$$

$$\frac{\partial^2 \hat{\psi}_i}{\partial y^2} + \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left(\bar{\rho}\frac{f_0^2}{N^2}\frac{\partial \hat{\psi}_i}{\partial z}\right) - \left(k^2 + k\delta_r\frac{\partial \bar{\pi}}{\partial y}\right)\hat{\psi}_i + k\delta_i\frac{\partial \bar{\pi}}{\partial y}\hat{\psi}_r = 0 \qquad (6.409)$$

where the Fourier transform of the streamfunction has been decomposed into

$$\hat{\psi} = \hat{\psi}_r + i\hat{\psi}_i \qquad (6.410)$$

and we have

$$\delta_r = \frac{\omega_r - k\bar{u}}{(\omega_r - k\bar{u})^2 + \Gamma^2} \qquad (6.411)$$

$$\delta_i = \frac{\Gamma}{(\omega_r - k\bar{u})^2 + \Gamma^2} \qquad (6.412)$$

Moreover, real and imaginary part of (6.406)/$(\omega - k\bar{u})$ are

$$\frac{\partial \hat{\psi}_r}{\partial z} + k\frac{\partial \bar{u}}{\partial z}\left(\delta_r\hat{\psi}_r + \delta_i\hat{\psi}_i\right) = 0 \qquad (6.413)$$

$$\frac{\partial \hat{\psi}_i}{\partial z} + k\frac{\partial \bar{u}}{\partial z}\left(\delta_r\hat{\psi}_i - \delta_i\hat{\psi}_r\right) = 0 \qquad (6.414)$$

Now we form $\hat{\psi}_i$ (6.408) $-\hat{\psi}_r$ (6.409) and multiply this by $\overline{\rho}$, with the result

$$\overline{\rho}\frac{\partial}{\partial y}\left(\hat{\psi}_i\frac{\partial \hat{\psi}_r}{\partial y} - \hat{\psi}_r\frac{\partial \hat{\psi}_i}{\partial y}\right) + \frac{\partial}{\partial z}\left[\overline{\rho}\frac{f_0^2}{N^2}\left(\hat{\psi}_i\frac{\partial \hat{\psi}_r}{\partial z} - \hat{\psi}_r\frac{\partial \hat{\psi}_i}{\partial z}\right)\right] - \overline{\rho}k\delta_i\frac{\partial \bar{\pi}}{\partial y}\left|\hat{\psi}\right|^2 = 0 \qquad (6.415)$$

This we integrate in y and z and obtain

$$\int_0^\infty dz\overline{\rho}\left[\hat{\psi}_i\frac{\partial \hat{\psi}_r}{\partial y} - \hat{\psi}_r\frac{\partial \hat{\psi}_i}{\partial y}\right]_0^{L_y} + \int_0^{L_y} dy\left[\overline{\rho}\frac{f_0^2}{N^2}\left(\hat{\psi}_i\frac{\partial \hat{\psi}_r}{\partial z} - \hat{\psi}_r\frac{\partial \hat{\psi}_i}{\partial z}\right)\right]_0^\infty$$

$$= \int_0^\infty dz\int_0^{L_y} dy\overline{\rho}k\delta_i\frac{\partial \bar{\pi}}{\partial y}\left|\hat{\psi}\right|^2 \qquad (6.416)$$

Application of the boundary conditions (6.405) and (6.208) yields

$$\int_0^\infty dz \int_0^{L_y} dy \overline{\rho} k \delta_i \frac{\partial \overline{\pi}}{\partial y} \left|\hat{\psi}\right|^2 + \int_0^{L_y} dy \left[\overline{\rho} \frac{f_0^2}{N^2} \left(\hat{\psi}_i \frac{\partial \hat{\psi}_r}{\partial z} - \hat{\psi}_r \frac{\partial \hat{\psi}_i}{\partial z}\right)\right]_{z=0} = 0 \tag{6.417}$$

From (6.413) and (6.414) we furthermore obtain

$$\hat{\psi}_i \frac{\partial \hat{\psi}_r}{\partial z} - \hat{\psi}_r \frac{\partial \hat{\psi}_i}{\partial z} = -k \frac{\partial \overline{u}}{\partial z} \delta_i \left|\hat{\psi}\right|^2 \tag{6.418}$$

so that (6.417) becomes, using (6.412),

$$\Gamma k \left[\int_0^\infty dz \overline{\rho} \int_0^{L_y} dy \frac{\partial \overline{\pi}}{\partial y} \frac{\left|\hat{\psi}\right|^2}{|\omega - k\overline{u}|^2} - \int_0^{L_y} dy \left(\overline{\rho} \frac{f_0^2}{N^2} \frac{\partial \overline{u}}{\partial z} \frac{\left|\hat{\psi}\right|^2}{|\omega - k\overline{u}|^2}\right)_{z=0}\right] = 0 \tag{6.419}$$

In the case of an instability, however, one has $\Gamma > 0$ so that we obtain as condition for this

$$\int_0^\infty dz \overline{\rho} \int_0^{L_y} dy \frac{\partial \overline{\pi}}{\partial y} \frac{\left|\hat{\psi}\right|^2}{|\omega - k\overline{u}|^2} - \int_0^{L_y} dy \left(\overline{\rho} \frac{f_0^2}{N^2} \frac{\partial \overline{u}}{\partial z} \frac{\left|\hat{\psi}\right|^2}{|\omega - k\overline{u}|^2}\right)_{z=0} = 0 \tag{6.420}$$

In the application of this three interesting cases must be considered:

No Wind Shear at the Ground ($\partial \overline{u}/\partial z|_{z=0} = 0$): In this case the second integral vanishes. For the other to also do so, $\partial \overline{\pi}/\partial y$ must change sign in the model volume. This is the *Rayleigh condition*, also to be used as condition for a *barotropic instability*.

Positive Meridional Gradient of Basic-Flow Potential Vorticity ($\partial \overline{\pi}/\partial y \geq 0$): This is the typical case, since typically planetary vorticity dominates so that

$$\frac{\partial \overline{\pi}}{\partial y} \approx \beta > 0 \tag{6.421}$$

Clearly, in this case we must have at least locally

$$\left.\frac{\partial \overline{u}}{\partial z}\right|_{z=0} > 0 \tag{6.422}$$

This is the scenario of a baroclinic instability.

Negative Wind Shear Everywhere at the Ground ($\partial\bar{u}/\partial z|_{z=0} < 0$)**:** This requires that at least locally

$$\frac{\partial\bar{\pi}}{\partial y} < 0 \tag{6.423}$$

For the reasons given above this is rather seldom.

The Eady Problem

Finally we consider an approximation, due to Eady (1949), enabling an analytical treatment of baroclinic waves and their growth rates. Essential aspects of baroclinic instability in a continuously stratified atmosphere are captured this way. The assumptions are:

- Density and stability of the reference atmosphere are constants, i.e.,

$$\overline{\rho} = \text{const.} \tag{6.424}$$

$$N^2 = \text{const.} \tag{6.425}$$

- We are on an f-plane, i.e.,

$$\beta = 0 \tag{6.426}$$

  The consequences of this approximation we could already discuss within the framework of a two-layer model.
- The zonal wind increases at a constant rate from the ground, i.e.,

$$\bar{u} = \Lambda z \tag{6.427}$$

  so that

$$\frac{\partial\bar{u}}{\partial z} = \Lambda = \text{const.} \tag{6.428}$$

- The atmosphere has solid boundaries at $z = 0$ and $z = H$. This means that (6.406) also holds at $z = H$.

Under these conditions one has

$$\frac{\partial\bar{\pi}}{\partial y} = 0 \tag{6.429}$$

so that (6.404) becomes

$$-k^2\hat{\psi} + \frac{\partial^2\hat{\psi}}{\partial y^2} + \frac{f_0^2}{N^2}\frac{\partial^2\hat{\psi}}{\partial z^2} = 0 \tag{6.430}$$

The boundary condition (6.406) is, together with its analogue for $z = H$

$$(\omega - k\Lambda z)\frac{\partial\hat{\psi}}{\partial z} + k\Lambda\hat{\psi} = 0 \qquad (z = 0, H) \tag{6.431}$$

The meridional boundary conditions (6.405), however, also imply

$$\hat{\psi}\,(k, y, z, \omega) = \sum_l \sin\,(l y)\,\tilde{\psi}\,(k, l, z, \omega) \tag{6.432}$$

$$l = m\frac{\pi}{L_y} \qquad (m \geq 1) \tag{6.433}$$

so that we obtain from (6.430)

$$\frac{\partial^2 \tilde{\psi}}{\partial z^2} - \alpha^2 \tilde{\psi} = 0 \tag{6.434}$$

with

$$\alpha^2 = \frac{N^2}{f_0^2}\left(k^2 + l^2\right) \tag{6.435}$$

The general solution of this equation is

$$\tilde{\psi} = A \sinh \alpha z + B \cosh \alpha z \tag{6.436}$$

Dispersion relation and structure of the corresponding baroclinic waves are obtained from the two boundary conditions (6.431). These yield

$$M \begin{pmatrix} A \\ B \end{pmatrix} = 0 \tag{6.437}$$

with

$$M = \begin{pmatrix} \omega\alpha & k\Lambda \\ \begin{matrix} \alpha\,(\omega - k\Lambda H)\cosh\alpha H \\ +k\Lambda \sinh\alpha H \end{matrix} & \begin{matrix} \alpha\,(\omega - k\Lambda H)\sinh\alpha H \\ +k\Lambda\cosh\alpha H \end{matrix} \end{pmatrix} \tag{6.438}$$

Non-trivial solutions only exist if

$$\det\,(M) = 0 \tag{6.439}$$

This leads to the dispersion relation

$$\boxed{\omega_{1,2} = k\frac{\Lambda H}{2} \pm k\frac{\Lambda H}{2}\sqrt{1 - \frac{4\cosh\alpha H}{\alpha H \sinh\alpha H} + \frac{4}{\alpha^2 H^2}}} \tag{6.440}$$

One has an instability if the argument of the square root is negative. Via

$$\tanh\alpha H = \frac{2\tanh\dfrac{\alpha H}{2}}{1 + \tanh^2\dfrac{\alpha H}{2}} \tag{6.441}$$

we find that this means that

$$
\begin{aligned}
0 &> \frac{4}{\alpha^2 H^2}\left(\frac{\alpha^2 H^2}{4} - \frac{\alpha H \cosh \alpha H}{\sinh \alpha H} + 1\right) \\
&= \frac{4}{\alpha^2 H^2}\left[\frac{\alpha^2 H^2}{4} - \frac{\alpha H}{2}\left(\coth \frac{\alpha H}{2} + \tanh \frac{\alpha H}{2}\right) + 1\right] \\
&= \frac{4}{\alpha^2 H^2}\left(\frac{\alpha H}{2} - \coth \frac{\alpha H}{2}\right)\left(\frac{\alpha H}{2} - \tanh \frac{\alpha H}{2}\right)
\end{aligned} \tag{6.442}
$$

The last factor is always positive, so that the instability condition is

$$\frac{\alpha H}{2} < \coth \frac{\alpha H}{2} \tag{6.443}$$

Numerically this leads to

$$\alpha H < 2.399 \tag{6.444}$$

or

$$\boxed{\sqrt{k^2 + l^2} < \frac{2.399}{L_{di}}} \tag{6.445}$$

The corresponding wavenumbers thus must satisfy

$$\lambda = \frac{2\pi}{\sqrt{k^2 + l^2}} > \frac{2\pi}{2.399} L_{di} \approx 3000\,\mathrm{km} \tag{6.446}$$

since in midlatitudes one has $L_{di} \approx 1000\,\mathrm{km}$. A further numerical analysis shows that the growth rate

$$\Gamma = k \frac{\Lambda H}{2} \sqrt{\frac{4 \cosh \alpha H}{\alpha H \sinh \alpha H} - 1 - \frac{4}{\alpha^2 H^2}} \tag{6.447}$$

is largest at

$$l = 0 \tag{6.448}$$

$$\alpha H = 1.6 \tag{6.449}$$

This leads to

$$\boxed{(k, l) = (1.6/L_{di}, 0)} \tag{6.450}$$

The corresponding zonal wavelength is

$$\lambda = \frac{2\pi}{1.6} L_{di} \approx 4000\,\mathrm{km} \tag{6.451}$$

There one has

$$\boxed{\Gamma \approx 0.3 \frac{\Lambda H}{L_{di}}} \tag{6.452}$$

With $\Lambda H = \bar{u}(z = H) \approx 20\,\text{m/s}$ and $L_{di} \approx 1000\,\text{km}$ this leads to $\Gamma \approx 0.52\text{d}^{-1}$. Figure 6.11 shows the complete dependence of the growth rate on the two horizontal wavenumbers.

The *vertical structure* can again be obtained from the vertical boundary conditions. (6.437) and (6.438) yield

$$A = -\frac{k\Lambda}{\alpha\omega} B = -\frac{k\Lambda}{\alpha}\left(\frac{\omega_r}{|\omega|^2} - i\frac{\Gamma}{|\omega|^2}\right) B \tag{6.453}$$

Inserting this into (6.436), with $B = A_\psi$, leads to

$$\tilde{\psi} = A_\psi \left(\cosh \alpha z - \frac{k\Lambda\omega_r}{\alpha\,|\omega|^2} \sinh \alpha z + i\frac{k\Lambda\Gamma}{\alpha\,|\omega|^2} \sinh \alpha z\right) \tag{6.454}$$

Then the tangent of the phase in $\tilde{\psi} = |\tilde{\psi}|e^{i\epsilon}$ is

$$\tan \epsilon = \frac{k\Lambda\Gamma}{\alpha\,|\omega|^2} \frac{\sinh \alpha z}{\cosh \alpha z - \dfrac{k\Lambda\omega_r}{\alpha\,|\omega|^2} \sinh \alpha z} \tag{6.455}$$

and its vertical derivative

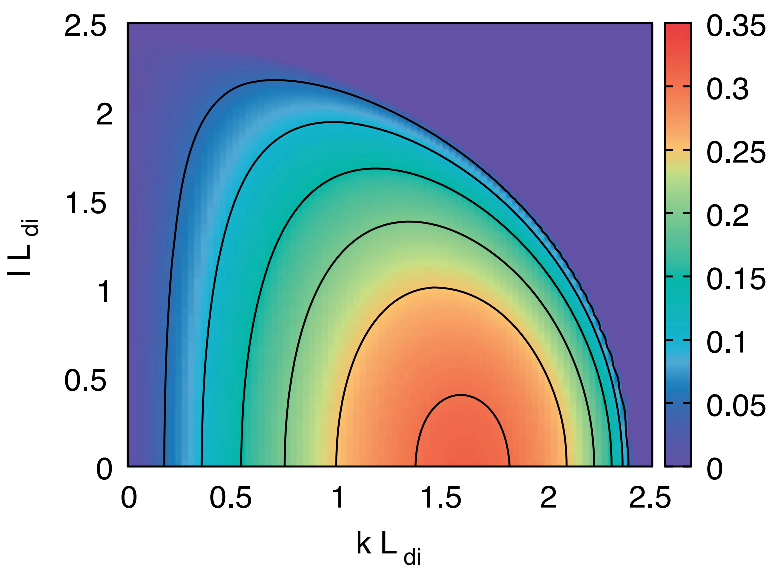

Fig. 6.11 Dependence of the growth rate Γ of a baroclinic wave within the framework of the Eady problem on zonal wavenumber k and meridional wavenumber l. Γ has been normalized by $\Lambda H/L_{di}$, and the wavenumbers by $1/L_{di}$

$$\frac{\partial}{\partial z}\tan\epsilon = \frac{k\Lambda\Gamma}{\alpha\,|\omega|^2}\,\frac{1}{\left(\cosh\alpha z - \dfrac{k\Lambda\omega_r}{\alpha\,|\omega|^2}\sinh\alpha z\right)^2} \tag{6.456}$$

For $\Gamma > 0$ the phase increases with altitude, and we again obtain the westward tilt of the growing wave with altitude. Correspondingly, a decaying wave exhibits an eastward phase tilt. Figure 6.12 shows the longitude–altitude structure of the wave. One clearly recognizes the westward phase tilt, and the phase equality between potential temperature and meridional wind in middle altitudes. Conspicuous is also the relatively simple altitude dependence. This is the reason why the two-layer model is so good at the description of the baroclinic instability.

6.4.3 Summary

The engine of daily weather at midlatitudes is baroclinic instability. Solar radiation establishes a *meridional potential-temperature gradient* between tropics and polar regions,

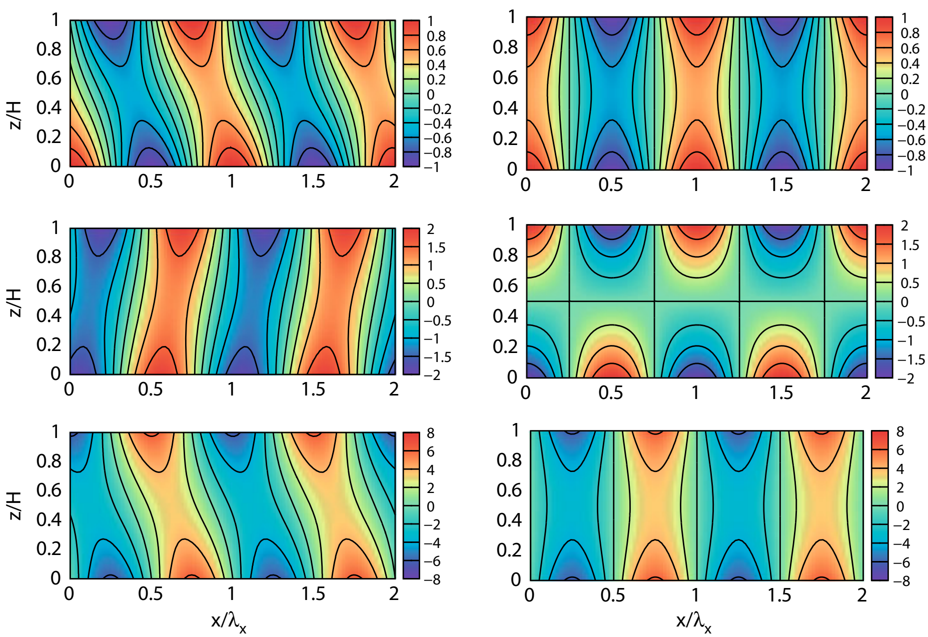

Fig. 6.12 Longitude–altitude structure of the most rapidly growing baroclinic wave *(left column)* and a marginally stable wave with $\Gamma \approx 0$ *(right)*. Shown are, in normalized quantities, the streamfunction ψ' *(top row)*, its vertical gradient $\partial\psi/\partial z$, again proportional to the potential temperature *(middle)*, and the meridional wind $v' = \partial\psi'/\partial x$ *(bottom)*

accompanied via the thermal-wind equation by a pronounced *jet stream* of strong westerlies in the upper troposphere. To these gradients the atmosphere reacts by the formation of *baroclinic waves working by a poleward heat transport against the origin of the instability*.

Essential aspects can already be recognized within the framework of the *two-layer model*:

- Again one considers the *linear dynamics* of infinitesimally small perturbations of the zonally-symmetric solution. Via Fourier transformation in zonal and meridional direction the *dynamics of separate wavenumber combinations are decoupled from each other*.
- In the solution of the *initial-value problem* the various *eigenmodes*, from which the general solution can be constructed, are not orthogonal any more, as was the case in the geostrophic adjustment problem in the shallow-water equations. Via the simultaneous solution of the corresponding *adjoint problem*, however, a decomposition of the initial state into the contributions from the various eigenmodes becomes possible.
- The eigenfrequencies result from the *dispersion relation*, identifying all combinations of wavenumber and frequency under which the linear operator of the problem becomes singular. The *polarization relations* yield the structure of the corresponding eigenmodes.
- *Neglect of the β-effect* yields the following results:
  - The atmosphere is *unstable* for every zonal-wind shear, i.e., there are eigenfrequencies with positive imaginary part, also known as *growth rate*.
  - This instability, however, only acts at *horizontal wavelengths above a lower threshold determined via κ by stratification and rotation*. At typical atmospheric conditions it is at about 3000 km.
  - The *most strongly growing baroclinic wave* has a purely *zonal propagation direction* in the horizontal.
  - The vertical structure exhibits a characteristic *westward phase inclination*. This implies that the growing baroclinic waves transport potential temperature from the tropics to the poles.
- A generalized analysis shows that the *β-effect stabilizes the flow*. For the possibility of an instability the zonal-wind shear must be above a threshold rising with the meridional planetary-vorticity gradient. Other determining factors are stratification and rotation. As a result of this no baroclinic instability is possible in the tropics.
- The analysis of the *energetics* of the growth process bears out that *poleward heat transport* is responsible for the wave growth. Via this process available potential energy of the basic flow is transformed into available potential energy of the waves. By the *rising of warm air masses and the sinking of cold air masses* this available potential energy is transformed into kinetic energy in the waves.

The more general treatment for the *continuously stratified atmosphere* yields the following additional aspects:

- In addition to the *linearized conservation equation for quasigeostrophic potential vorticity* here also the *linearization of the vertical boundary conditions* must be taken into account, which had already been incorporated into the two-layer model equations. The vertical wind vanishes at the ground. Toward the top either density or vertical wind must vanish.
- The *Rayleigh theorem* yields necessary conditions for the instability of general zonally-symmetric profiles of zonal wind and potential temperature in thermal-wind balance. The most important cases are:
  - If the vertical zonal-wind shear vanishes at the ground the meridional potential-vorticity gradient must change sign in the atmosphere.
  - However, if the latter is positive everywhere, which is typically the case, there must be regions at the ground where the zonal wind increases vertically.
- An analytical solution of the stability problem is possible in the *Eady approximation*. One assumes that the atmosphere is homogeneous and neglects the β-effect.
  - One noteworthy result is that the shortest possible horizontal wavelength at which an instability can occur scales with the *internal Rossby deformation radius*. The same holds for the wavelength where the instability maximizes. Thus also here rotation and stratification control the instability scales.
  - As also in the two-layer model the *largest growth rate* scales with the *ratio between upper-troposphere zonal wind and internal Rossby deformation radius*.
  - The westward phase tilt is also found. In general the vertical structure is simple enough to explain why the two-layer model can give results as realistic.

6.5 Recommendations for Further Reading

Recommendable textbooks, each covering the topic of quasigeostrophic theory and baroclinic instability in its own way, are Holton and Hakim (2013), Pedlosky (1987), Salmon (1998), and Vallis (2006). The first development of quasigeostrophic theory has been done by Charney (1948), while two-layer models go back to Phillips (1954). Although one often sees quasigeostrophic theory being used in studies of large-scale planetary waves, strictly speaking it does not apply to those, because the horizontal scale is not smaller than the radius of earth anymore. Phillips (1963) has suggested instead the planetary geostrophic equations, and the reader might consult Dolaptchiev and Klein (2013), Dolaptchiev et al. (2019), and the sources therein on the present state of this topic. Classic books on all kinds of hydrodynamic instabilities are Chandrasekhar (1981) and Drazin and Reid (2004). The original source for the Eady model is Eady (1949). A treatment of baroclinic instability with less restrictions on the assumed atmosphere has been given by Charney (1947). Classic papers on the nonlinear development of baroclinic instability are Simmons and Hoskins (1978), and a thorough discussion of the topic from the potential-vorticity perspective has been given by Hoskins et al. (1985). Original citations on the Rayleigh theorem applied to the atmosphere are Charney and Stern (1962) and Pedlosky (1964).