8 The Interaction Between Rossby Waves and the Mean Flow

The following questions are central in studies of the global circulation:

- Is the mean flow influenced by atmospheric waves like gravity waves or Rossby waves? And if so, how?
- Vice versa, how does the mean flow influence the waves?

Here, the term *mean flow* stands for the zonal average, which is for an arbitrary quantity X defined as

$$\langle X \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\lambda X \tag{8.1}$$

It often suffices to consider the β-channel. Here one would correspondingly use

$$\langle X \rangle = \frac{1}{L_x} \int_0^{L_x} dx X \tag{8.2}$$

where L_x is the distance in x over which all quantities are periodic, so that $X(x) = X(x+L_x)$. *Eddies* are deviations

$$X' = X - \langle X \rangle \tag{8.3}$$

from the mean flow. A *wave* is an eddy that satisfies a dispersion relation. Because dispersion relations are a result of linear theory, this requires the amplitude of the waves to be sufficiently small. Classic turbulence is not composed from waves, but atmospheric variability can often be seen as resulting from a superposition of waves. This does not rule out nonlinear interactions between the individual waves. In the next two chapters we first discuss the basics of the interaction between waves and a mean flow, which will then be applied to the problem

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of the meridional circulation. For the time being we will leave out the topic of gravity waves. In most cases it is useful to resort to quasigeostrophic theory which will be given ample space.

8.1 Basics of Quasigeostrophic Theory

For better readability we begin by recapitulating the most important elements of quasi-geostrophic theory. First applications and extensions lay the ground for the subsequent discussions. For reasons of simplicity, but without loss of generality, we generally restrict ourselves to the geometry of the β-plane.

8.1.1 The Governing Equations

In quasigeostrophic theory the variable part of all quantities can be obtained from the streamfunction

$$\psi = \frac{p - \overline{p}}{f_0 \overline{\rho}} \tag{8.4}$$

with p being the pressure, and $\overline{p}$ and $\overline{\rho}$ pressure and density of the reference atmosphere, which only depends on altitude. In particular the geostrophic horizontal wind can be computed from

$$u_g = -\frac{\partial \psi}{\partial y} \tag{8.5}$$

$$v_g = \frac{\partial \psi}{\partial x} \tag{8.6}$$

The potential temperature is

$$\theta = \overline{\theta} \left(1 + \frac{f_0}{g} \frac{\partial \psi}{\partial z} \right) \tag{8.7}$$

Here $\overline{\theta}$ is the potential temperature of the reference atmosphere. Defining $\tilde{\theta} = \theta - \overline{\theta}$ as the deviation of the potential temperature from $\overline{\theta}$, buoyancy becomes

$$b = g \frac{\tilde{\theta}}{\overline{\theta}} \tag{8.8}$$

or

$$b = f_0 \frac{\partial \psi}{\partial z} \tag{8.9}$$

Quasigeostrophic potential vorticity

$$\pi = \zeta + f + \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \left(\overline{\rho} \frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right) \tag{8.10}$$

therefore can be written as

$$\pi = \zeta + f + \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \left(\overline{\rho} \frac{f_0}{N^2} b \right) \tag{8.11}$$

where

$$\zeta = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \nabla_h^2 \psi \tag{8.12}$$

is the relative vorticity and

$$f = f_0 + \beta y \tag{8.13}$$

the latitude-dependent planetary vorticity. $\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ denotes the horizontal part of the Laplacian. Potential vorticity satisfies the conservation equation

$$\frac{\partial \pi}{\partial t} + \mathbf{u}_g \cdot \nabla \pi = D \tag{8.14}$$

with D describing the influence of all non-conservative processes. This includes friction, heating, and heat conduction. Using (8.5) and (8.6) the conservation equation can be rewritten as

$$\frac{\partial \pi}{\partial t} + J(\psi, \pi) = D \tag{8.15}$$

with

$$J(A, B) = \frac{\partial A}{\partial x} \frac{\partial B}{\partial y} - \frac{\partial A}{\partial y} \frac{\partial B}{\partial x} \tag{8.16}$$

the Jacobi operator for arbitrary fields A and B. In a similar manner the quasigeostrophic entropy equation can be written as

$$\frac{\partial \tilde{\theta}}{\partial t} + J(\psi, \tilde{\theta}) + w \frac{d\overline{\theta}}{dz} = q_\theta \tag{8.17}$$

Here all non-conservative terms are subsumed under q_θ. Using (8.8) the entropy equation becomes

$$\frac{\partial b}{\partial t} + J(\psi, b) + w N^2 = Q \tag{8.18}$$

where

$$N^2 = \frac{g}{\overline{\theta}} \frac{d\overline{\theta}}{dz} \tag{8.19}$$

is the squared Brunt–Väisälä frequency, and $Q = gq_\theta/\overline{\theta}$. The lower boundary condition from Ekman theory prescribes

$$z = 0: \qquad w = \sqrt{\frac{K}{2f_0}}\nabla_h^2\psi \tag{8.20}$$

or, using (8.18),

$$z = 0: \qquad \frac{\partial b}{\partial t} + J(\psi, b) = -N^2\sqrt{\frac{K}{2f_0}}\nabla_h^2\psi + Q \tag{8.21}$$

8.1.2 Conservation Properties

Under the condition

$$D = Q = K = 0 \tag{8.22}$$

quasigeostrophic dynamics conserves various quantities. One of those is the *total energy*

$$E = \frac{1}{2}\int_V dV\overline{\rho}\left(|\nabla_h\psi|^2 + \frac{b^2}{N^2}\right) \tag{8.23}$$

where one must integrate over the volume of the entire atmosphere. One has

$$\frac{dE}{dt} = 0 \tag{8.24}$$

Moreover, because of (8.14) every differentiable function $F(\pi)$ satisfies

$$\frac{\partial F}{\partial t} + \mathbf{u}_g \cdot \nabla F = 0 \tag{8.25}$$

Due to the fact that the geostrophic wind has no horizontal divergence

$$\nabla \cdot \mathbf{u}_g = 0 \tag{8.26}$$

we can also write

$$\frac{\partial F}{\partial t} + \nabla \cdot \left(\mathbf{u}_g F\right) = 0 \tag{8.27}$$

Given regular boundary conditions, like the ones of a β-channel, the integral of the divergence term over the entire surface area S of the atmosphere at a fixed altitude z vanishes, by applying Gauss's theorem. Hence one obtains for every altitude

$$\frac{d}{dt}\int_S dS\, F = 0 \tag{8.28}$$

In particular, *enstrophy*

$$Z = \int_S dS \, \frac{\pi^2}{2} \tag{8.29}$$

is also a conserved quantity.

8.1.3 The Quasigeostrophic Enstrophy Equation Within Linear Dynamics

At this point we begin differentiating between zonal mean and eddies (or waves). We decompose streamfunction and potential vorticity via

$$\psi = \langle\psi\rangle + \psi' \tag{8.30}$$

$$\pi = \langle\pi\rangle + \pi' \tag{8.31}$$

where we assume that the wave part is small. Inserting this into the conservation equation (8.14) we first obtain

$$\frac{\partial\langle\pi\rangle}{\partial t} + \frac{\partial\pi'}{\partial t} + \langle u_g\rangle\frac{\partial\pi'}{\partial x} + u'_g\frac{\partial\pi'}{\partial x} + \langle v_g\rangle\frac{\partial\langle\pi\rangle}{\partial y} + \langle v_g\rangle\frac{\partial\pi'}{\partial y} + v'_g\frac{\partial\langle\pi\rangle}{\partial y} + v'_g\frac{\partial\pi'}{\partial y} = \langle D\rangle + D' \tag{8.32}$$

From (8.6) and the periodic boundary conditions in x we find

$$\langle v_g\rangle = 0 \tag{8.33}$$

so that

$$\frac{\partial\langle\pi\rangle}{\partial t} + \frac{\partial\pi'}{\partial t} + \langle u_g\rangle\frac{\partial\pi'}{\partial x} + v'_g\frac{\partial\langle\pi\rangle}{\partial y} + u'_g\frac{\partial\pi'}{\partial x} + v'_g\frac{\partial\pi'}{\partial y} = \langle D\rangle + D' \tag{8.34}$$

Averaging this equation yields

$$\frac{\partial\langle\pi\rangle}{\partial t} + \left\langle u'_g\frac{\partial\pi'}{\partial x} + v'_g\frac{\partial\pi'}{\partial y}\right\rangle = \langle D\rangle \tag{8.35}$$

Taking the difference (8.34)–(8.35) and neglecting all nonlinear terms in the wave quantities yields

$$\frac{\partial\pi'}{\partial t} + \langle u_g\rangle\frac{\partial\pi'}{\partial x} + v'_g\frac{\partial\langle\pi\rangle}{\partial y} = D' \tag{8.36}$$

Multiplying by π', and averaging afterward, leads to the enstrophy equation

$$\boxed{\frac{\partial}{\partial t}\left\langle\frac{\pi'^2}{2}\right\rangle = -\langle v'_g\pi'\rangle\frac{\partial\langle\pi\rangle}{\partial y} + \langle\pi'D'\rangle} \tag{8.37}$$

Here we have used

$$\left\langle \pi' \frac{\partial \pi'}{\partial x} \right\rangle = \left\langle \frac{\partial}{\partial x} \frac{\pi'^2}{2} \right\rangle = 0 \tag{8.38}$$

Obviously the density of enstrophy $\pi'^2/2$ is a measure of wave activity. Thus the enstrophy equation has two important consequences:

- Typically $\langle \pi' D' \rangle$ is a sink. Therefore growth of waves is only possible if

$$\langle v_g' \pi' \rangle \frac{\partial \langle \pi \rangle}{\partial y} < 0 \tag{8.39}$$

  Hence, the meridional flux of the potential vorticity must be directed *against* the meridional gradient of mean potential vorticity.
- In the absence of non-conservative processes, i.e., if $D' = 0$, the wave amplitude is only steady if

$$\langle v_g' \pi' \rangle = 0, \tag{8.40}$$

  i.e., if the flux of the potential vorticity vanishes. Only when the mean potential vorticity is independent of latitude, this is not necessary. Due to the β-term in the latitudinal gradient of the potential vorticity that is very unlikely, however.

8.1.4 Summary

In quasigeostrophic theory, not only energy and the integral of quasigeostrophic potential vorticity π are conserved, but also the integral of arbitrary functions of π. An important example is *enstrophy,* with its density $\pi^2/2$. The enstrophy of *eddies,* i.e., deviations from the zonal mean, is a measure of their amplitude. Within the linear approximation, holding at weak amplitudes, *eddy enstrophy only changes if non-conservative processes act or if there is a meridional flux of potential vorticity.* It increases (decreases) eddy enstrophy when it is directed against (into) the direction of the meridional gradient of zonal-mean quasigeostrophic potential vorticity .

8.2 Rossby-Wave Propagation

A very useful tool for the conceptional understanding of wave dynamics is WKB theory that will first be derived here in a heuristic manner. It then finds its first application in a discussion of Rossby-wave propagation into the middle atmosphere.

8.2.1 Wave Propagation Within WKB Theory

If

- the wave amplitude is small enough and furthermore
- the mean fields are changing only slowly in space and time,

the ray theory of Wentzel, Kramers, and Brillouin provides a solution of the linear equation (8.36) with $D' = 0$

$$\frac{\partial \pi'}{\partial t} + \langle u_g \rangle \frac{\partial \pi'}{\partial x} + v_g' \frac{\partial \langle \pi \rangle}{\partial y} = 0 \tag{8.41}$$

with far-reaching consequences for the interpretation of wave dynamics.

$$\pi' = \nabla_h^2 \psi' + \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \left(\frac{\overline{\rho} f_0^2}{N^2} \frac{\partial \psi'}{\partial z} \right) \tag{8.42}$$

is the potential vorticity of the waves and

$$\langle \pi \rangle = \frac{\partial^2 \langle \psi \rangle}{\partial y^2} + f + \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \left(\frac{\overline{\rho} f_0^2}{N^2} \frac{\partial \langle \psi \rangle}{\partial z} \right) \tag{8.43}$$

the potential vorticity of the zonal mean. A general solution of (8.41) is difficult, because $\overline{\rho}$ and N^2 as well as $\langle \psi \rangle$ can have complicated space-time dependencies. In order to motivate the solution within WKB theory we recall the properties of a Rossby wave in an isothermal hydrostatic atmosphere with constant mean flow $\langle u_g \rangle$. In this case one obtains the stream-function

$$\psi' = \hat{\psi}\, e^{z/2H + i(kx + ly + mz - \omega t)} \tag{8.44}$$

with the dispersion relation

$$\omega = k \langle u_g \rangle - \frac{\beta k}{k^2 + l^2 + \frac{f_0^2}{N^2} m^2 + \frac{1}{4 L_{di}^2}} \tag{8.45}$$

Here

$$H = \frac{R\overline{T}}{g} \tag{8.46}$$

is the constant scale height of the reference atmosphere. The corresponding profiles of density and pressure are

$$\overline{\rho} = \rho_0 e^{-z/H} \tag{8.47}$$

$$\overline{p} = p_0 e^{-z/H} \tag{8.48}$$

with fixed ground-level values ρ_0 and p_0. All parameters in the dispersion relation are constants: Using (8.19) and (8.48)

$$\overline{\theta} = \overline{T}\left(\frac{p_{00}}{\overline{p}}\right)^{R/c_p} \tag{8.49}$$

yields

$$N^2 = \frac{g}{H_\theta} \tag{8.50}$$

with

$$\frac{1}{H_\theta} = \frac{R}{c_p}\frac{1}{H} \tag{8.51}$$

the inverse scale height of potential temperature. Furthermore

$$L_{di} = \frac{NH}{f_0} \tag{8.52}$$

is the internal Rossby deformation radius, which is constant as well, and

$$\beta = \frac{\partial\langle\pi\rangle}{\partial y} \tag{8.53}$$

is the constant gradient of the mean potential vorticity. The essential in these results is that the parameters $\langle u_g\rangle$, β, H, and N^2 are constants and that the streamfunction is

$$\psi' \propto \frac{1}{\sqrt{\overline{\rho}}} \tag{8.54}$$

That is, it is inversely proportional to the square root of the reference density.

Motivated by the last insight we now use the ansatz

$$\psi' = \frac{\tilde{\psi}}{\sqrt{\overline{\rho}}} \tag{8.55}$$

This implies no constraint, because we do not constrain $\tilde{\psi}$. One then has

$$\frac{\partial\psi'}{\partial z} = \frac{1}{\sqrt{\overline{\rho}}}\left(\frac{\partial\tilde{\psi}}{\partial z} + \frac{\tilde{\psi}}{2H}\right) \tag{8.56}$$

where

$$\frac{1}{H} = -\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dz} \tag{8.57}$$

is the inverse scale height of the reference density, given in the isothermal case by (8.46). Moreover, one obtains

$$
\begin{aligned}
&\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\frac{\overline{\rho} f_0^2}{N^2}\frac{\partial \psi'}{\partial z}\right) \\
&= \frac{1}{\sqrt{\overline{\rho}}}\left\{\frac{d}{dz}\left(\frac{f_0^2}{N^2}\right)\left(\frac{\partial \tilde{\psi}}{\partial z} + \frac{\tilde{\psi}}{2H}\right) + \frac{f_0^2}{N^2}\left[\frac{\partial^2 \tilde{\psi}}{\partial z^2} - \frac{\tilde{\psi}}{4H^2} + \frac{\partial}{\partial z}\left(\frac{1}{2H}\right)\tilde{\psi}\right]\right\}
\end{aligned}
\tag{8.58}
$$

Under isothermal conditions the derivatives of H and N^2 vanish. Now we assume that these derivatives are small enough so that they can be neglected. More specifically, we assume that H and N^2 are varying weakly enough such that

$$
\left|N^2\frac{d}{dz}\left(\frac{1}{N^2}\right)\frac{\partial \tilde{\psi}}{\partial z}\right| = \left|\frac{1}{N^2}\frac{dN^2}{dz}\frac{\partial \tilde{\psi}}{\partial z}\right| \ll \left|\frac{\partial^2 \tilde{\psi}}{\partial z^2}\right| \tag{8.59}
$$

$$
\left|N^2\frac{d}{dz}\left(\frac{1}{N^2}\right)\right| = \left|\frac{1}{N^2}\frac{dN^2}{dz}\right| \ll \frac{1}{2H} \tag{8.60}
$$

$$
\left|H\frac{d}{dz}\left(\frac{1}{H}\right)\right| = \left|\frac{1}{H}\frac{dH}{dz}\right| \ll \frac{1}{2H} \tag{8.61}
$$

hold. The first condition implies that the vertical scale on which N^2 varies substantially is significantly larger than the vertical scale on which the streamfunction of the linear solution changes. Below we will see that

$$
\tilde{\psi} \propto e^{imz} \tag{8.62}
$$

Hence (8.59) is equivalent to

$$
\left|\frac{1}{N^2}\frac{dN^2}{dz}\right| \ll |m| \tag{8.63}
$$

From (8.60) and (8.61) one can see that N^2 and H only vary on a vertical scale which is significantly larger than $2H$. If (8.59)–(8.61) are satisfied, then (8.58) becomes

$$
\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\frac{\overline{\rho} f_0^2}{N^2}\frac{\partial \psi'}{\partial z}\right) \approx \frac{1}{\sqrt{\overline{\rho}}}\frac{f_0^2}{N^2}\left(\frac{\partial^2 \tilde{\psi}}{\partial z^2} - \frac{\tilde{\psi}}{4H^2}\right) \tag{8.64}
$$

and

$$
\pi' \approx \frac{\tilde{\pi}}{\sqrt{\overline{\rho}}} \tag{8.65}
$$

with

$$
\tilde{\pi} = \nabla_h^2 \tilde{\psi} + \frac{f_0^2}{N^2}\left(\frac{\partial^2 \tilde{\psi}}{\partial z^2} - \frac{\tilde{\psi}}{4H^2}\right) \tag{8.66}
$$

Inserting (8.55) into (8.41) and putting all together yield

$$
\left(\frac{\partial}{\partial t} + \langle u_g \rangle\right)\frac{\partial \tilde{\pi}}{\partial x} + \frac{\partial \tilde{\psi}}{\partial x}\frac{\partial \langle \pi \rangle}{\partial y} = 0 \tag{8.67}
$$

Up to this point we have achieved elimination of the altitude dependence of $\overline{\rho}$ and N^2 in the equation. This does not mean that we neglect it, however. We just assume that it is sufficiently smooth.

The remaining equation still contains the mean zonal wind and the latitudinal gradient of the mean potential vorticity. In the isothermal case discussed above these are constants, hence allowing plane waves as solutions. Now we assume that *the fields of the mean flow vary only negligibly over one wavelength and over one period of the wave solution.* WKB theory then uses the ansatz

$$\tilde{\psi} = A(\mathbf{x}, t)e^{i\alpha(\mathbf{x},t)} \tag{8.68}$$

with A being a complex amplitude and α a (real-valued) phase.[1] A plane wave has $A = \hat{\psi}$ and $\alpha = kx + ly + mz - \omega t$. Correspondingly, the *local wavenumber* is defined by

$$\mathbf{k}(\mathbf{x}, t) = \nabla\alpha \tag{8.69}$$

and the *local frequency* by

$$\omega(\mathbf{x}, t) = -\frac{\partial\alpha}{\partial t} \tag{8.70}$$

Another important assumption of WKB theory is that A, $\mathbf{k}$, *and* ω *are also varying only slightly over one wavelength or one period of the wave solution.* This means that

$$|\nabla A| \ll |\mathbf{k}A| \tag{8.71}$$

$$\left|\frac{\partial A}{\partial t}\right| \ll |\omega A| \tag{8.72}$$

$$|\nabla k_i| \ll |\mathbf{k}k_i| \tag{8.73}$$

$$\left|\frac{\partial k_i}{\partial t}\right| \ll |k_i\omega| \tag{8.74}$$

$$|\nabla\omega| \ll |\mathbf{k}\omega| \tag{8.75}$$

$$\left|\frac{\partial\omega}{\partial t}\right| \ll \left|\omega^2\right| \tag{8.76}$$

hold. Therefore one obtains, for example,

$$\nabla\tilde{\psi} = (\nabla A + i\mathbf{k}A)\, e^{i\alpha} \approx i\mathbf{k}Ae^{i\alpha} = i\mathbf{k}\tilde{\psi} \tag{8.77}$$

and thus also

$$\begin{aligned}\nabla_h^2\tilde{\psi} &\approx \nabla_h \cdot \left(i\mathbf{k}_h Ae^{i\alpha}\right) = \left[i\left(\nabla_h \cdot \mathbf{k}_h + \mathbf{k}_h \cdot \nabla_h\right) A - |\mathbf{k}_h|^2\, A\right] e^{i\alpha} \approx -\,|\mathbf{k}_h|^2\, Ae^{i\alpha} \\ &= -\,|\mathbf{k}_h|^2\,\tilde{\psi}\end{aligned} \tag{8.78}$$

[1] It is not required here that the streamfunction is real-valued. As we will see below, real-valued fields are obtained from superposing WKB solutions with opposite wavenumbers and with complex-conjugate amplitudes.

as well as

$$\frac{\partial^2 \tilde{\psi}}{\partial z^2} \approx -m^2 \tilde{\psi} \tag{8.79}$$

with $\mathbf{k}_h = k\mathbf{e}_x + l\mathbf{e}_y$ the horizontal part of the wavenumber vector. This way one obtains

$$\tilde{\pi} \approx -\left(k^2 + l^2 + \frac{f_0^2}{N^2} m^2 + \frac{1}{4L_{di}^2}\right) \tilde{\psi} \tag{8.80}$$

Applying the WKB rules one more time yields

$$\frac{\partial \tilde{\pi}}{\partial t} \approx -i\omega\tilde{\pi} \tag{8.81}$$

$$\frac{\partial \tilde{\pi}}{\partial x} \approx ik\tilde{\pi} \tag{8.82}$$

$$\frac{\partial \tilde{\psi}}{\partial x} \approx ik\tilde{\psi} \tag{8.83}$$

Inserting (8.80)–(8.83) into (8.67) finally yields

$$i\left(\omega - k\langle u_g\rangle\right)\left(k^2 + l^2 + \frac{f_0^2}{N^2} m^2 + \frac{1}{4L_{di}^2}\right) \tilde{\psi} + ik\tilde{\psi} \frac{\partial \langle \pi \rangle}{\partial y} = 0 \tag{8.84}$$

Therefore the dispersion relation for Rossby waves is

$$\boxed{\omega = k\langle u_g\rangle - \frac{k \dfrac{\partial \langle \pi \rangle}{\partial y}}{k^2 + l^2 + \dfrac{f_0^2}{N^2} m^2 + \dfrac{1}{4L_{di}^2}}} \tag{8.85}$$

Note that this is a relation between the *local* space-time frequency and the *local* wave number. Because the mean flow and the reference atmosphere are space dependent and the mean flow can even be time-dependent, this relation is also dependent on space and time. We can write it abstractly as

$$\omega = \Omega\left(\mathbf{k}, \mathbf{x}, t\right) \tag{8.86}$$

Here the *explicit* dependency on space and time is only present in the parameters $\langle u_g\rangle$, $\partial\langle\pi\rangle/\partial y$, N^2, and L_{di}^2, but not in the wave number!

We introduce formally the *group velocity*

$$\mathbf{c}_g = \nabla_k \Omega \tag{8.87}$$

or, component-wise,

$$c_{gi} = \frac{\partial \Omega}{\partial k_i} \tag{8.88}$$

From the dispersion relation, and (8.69) and (8.70), results

$$\frac{\partial k_i}{\partial t} = \frac{\partial^2 \alpha}{\partial t \partial x_i} = -\frac{\partial \omega}{\partial x_i} = -\frac{\partial \Omega}{\partial k_j}\frac{\partial k_j}{\partial x_i} - \frac{\partial \Omega}{\partial x_i} \tag{8.89}$$

Using the definition of the group velocity this yields a prognostic equation for the wave number

$$\boxed{\frac{\partial k_i}{\partial t} + \mathbf{c}_g \cdot \nabla k_i = -\frac{\partial \Omega}{\partial x_i}} \tag{8.90}$$

Additionally, we get in a similar manner

$$\frac{\partial \omega}{\partial t} = \frac{\partial \Omega}{\partial k_i}\frac{\partial k_i}{\partial t} + \frac{\partial \Omega}{\partial t} = -c_{gi}\frac{\partial \omega}{\partial x_i} + \frac{\partial \Omega}{\partial t} \tag{8.91}$$

which results in

$$\boxed{\frac{\partial \omega}{\partial t} + \mathbf{c}_g \cdot \nabla \omega = \frac{\partial \Omega}{\partial t}} \tag{8.92}$$

The *ray equations* (8.90) and (8.92) describe the change in the wavenumber and the frequency along a ray that moves with the local group velocity $\mathbf{c}_g$. These equations have interesting symmetry properties:

- If the mean flow is invariant in the ith spatial direction, then the corresponding wave number k_i does not vary along the ray. Hence the zonal wave number k is a constant.
- If the mean flow is constant in time, then the frequency along a ray does not vary.

Finally one should note that (8.41) is a linear equation. Individual solutions can be superimposed linearly and yield another solution. Therefore one can compose a complex wave field from individual WKB solutions which again solves (8.41). Because in the zonally symmetric mean flow the zonal wavenumber k of an individual solution is a constant, one can write

$$\psi' = \frac{1}{\sqrt{\overline{\rho}}}\sum_k \tilde{\psi}_k = \frac{1}{\sqrt{\overline{\rho}}}\sum_k A_k(\mathbf{x}, t)e^{i\alpha_k(\mathbf{x},t)} \tag{8.93}$$

where

$$k = n\frac{2\pi}{L_x} \qquad n \in \mathbb{Z} \tag{8.94}$$

must hold in order for the periodicity to be satisfied in x. Furthermore

$$\mathbf{k}_k = \nabla \alpha_k = k\mathbf{e}_x + l_k\mathbf{e}_y + m_k\mathbf{e}_z \tag{8.95}$$

$$\omega_k = -\frac{\alpha_k}{\partial t} \tag{8.96}$$

are the corresponding wavenumber vectors and frequencies, which vary in space and time, following to (8.90) and (8.92). The prediction of the amplitudes A_k is subject of the next section.

8.2.2 Rossby-Wave Propagation into the Stratosphere

The dispersion relation (8.85) and the corresponding ray equations (8.90) and (8.92) have important consequences for the ability of Rossby waves to propagate in the vertical. If applied to a zonal and climatological mean $\langle u\rangle(y,z)$ of the zonal wind, here identified with its geostrophic part $\langle u_g\rangle(y,z)$, the ray equations imply for along-ray variations of frequency and zonal wavenumber

$$\left(\frac{\partial}{\partial t} + \mathbf{c}_g \cdot \nabla\right)\omega = 0 \tag{8.97}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{c}_g \cdot \nabla\right)k = 0 \tag{8.98}$$

That is, they are constant, and thereby also the zonal phase velocity $c = \omega/k$. The meridional and vertical wavenumbers, however, do change, according to

$$\left(\frac{\partial}{\partial t} + \mathbf{c}_g \cdot \nabla\right)l = -k\left[\frac{\partial\langle u\rangle}{\partial y} - \frac{\dfrac{\partial^2\langle\pi\rangle}{\partial y^2}}{k^2 + l^2 + \dfrac{f_0^2}{N^2}\left(m^2 + \dfrac{1}{4H^2}\right)}\right] \tag{8.99}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathbf{c}_g \cdot \nabla\right)m = -k\Bigg\{&\frac{\partial\langle u\rangle}{\partial z} - \frac{\dfrac{\partial^2\langle\pi\rangle}{\partial y\partial z}}{k^2 + l^2 + \dfrac{f_0^2}{N^2}\left(m^2 + \dfrac{1}{4H^2}\right)}\\ &-\frac{\partial\langle\pi\rangle}{\partial y}\frac{f_0^2}{N^2}\frac{\dfrac{1}{N^2}\dfrac{dN^2}{dz}\left(m^2 + \dfrac{1}{4H^2}\right) + \dfrac{1}{2H^3}\dfrac{dH}{dz}}{\left[k^2 + l^2 + \dfrac{f_0^2}{N^2}\left(m^2 + \dfrac{1}{4H^2}\right)\right]^2}\Bigg\}\end{aligned} \tag{8.100}$$

where we have replaced $L_{di} = NH/f_0$. These equations can be integrated for typical zonal-wind climatologies, with much to learn from such an exercise. Hoskins and Karoly (1981) and Karoly and Hoskins (1982) show results from such calculations. Among other things they find that the meridional propagation is influenced essentially by the sphericity of the earth's atmosphere.

The essential results on vertical propagation, however, can best be understood by direct consideration of the dispersion relation, and realizing that due to $0 < m^2 < \infty$ the frequency at a given meridional wave number has upper and lower limits so that

$$k\left(\langle u\rangle - u_c\right) < \omega = ck < k\langle u\rangle \tag{8.101}$$

with

$$u_c = \frac{\dfrac{\partial\langle\pi\rangle}{\partial y}}{k^2 + l^2 + \dfrac{1}{4L_{di}^2}} \tag{8.102}$$

Here we assume $\partial\langle\pi\rangle/\partial y > 0$. This is no serious limitation since the β-term usually dominates the meridional derivative of the quasigeostrophic potential vorticity. Figure 8.1 shows the dependence on vertical wavenumber for $k > 0$, as can be assumed without loss of generality, together with the limits following from (8.101). The slope of the curve is the vertical group velocity; i.e., ray propagation is upward (downward) at positive (negative) wavenumber. It is important to realize that ray propagation is such that frequency $\omega = \omega_0$ does not change, but that due to ray propagation to different (y, z) the upper and lower bounds $k\langle u\rangle$ and $k(\langle u\rangle - u_c)$ do change. Hence a ray moves on the ω curve in such a way that $\omega = \omega_0$. This also implies a variation of vertical group velocity, leading to corresponding variations of those bounds. Two interesting cases occur:

Wave Reflection

Assume that a ray propagates upward, i.e., $m > 0$, and that the zonal-mean zonal wind increases with altitude, i.e., $\partial\langle u\rangle/\partial z > 0$, as is the case, e.g., in the winter stratosphere. Then the lower bound $k(\langle u\rangle - u_c)$ will typically increase so that m must decrease. In case it crosses the zero threshold $m = 0$, the vertical group velocity changes its sign and the ray

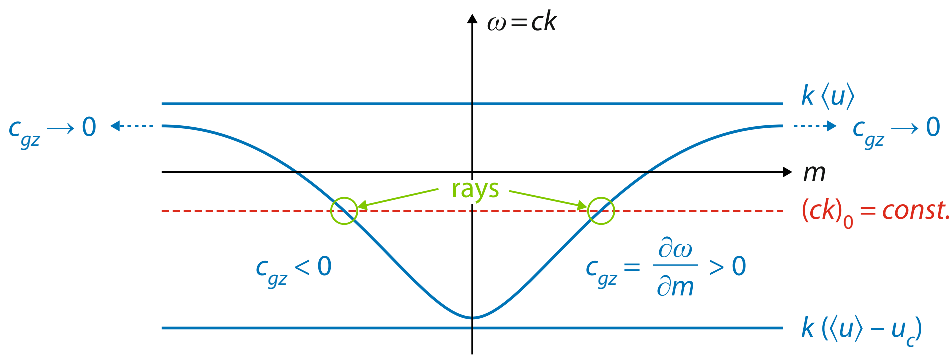

Fig. 8.1 Dependence of Rossby-wave frequency on vertical wavenumber, together with its extrema according to (8.101), assuming $k > 0$.

changes its direction of vertical propagation so that $k(\langle u\rangle - u_c)$ will begin to decrease. An analogous scenario would be downward propagation of a Rossby wave, with $m < 0$, into regions of increasing zonal-mean zonal wind, i.e., $\partial\langle u\rangle/\partial z < 0$, leading just the same way to a change in propagation direction at a reflection point where the zero threshold $m = 0$ is crossed.

Critical Lines

Assume now that a ray propagates upwards, i.e., $m > 0$, and that the zonal-mean zonal wind decreases with altitude, i.e. $\partial\langle u\rangle/\partial z < 0$, as is typical in the summer stratosphere. Then the upper bound $k\langle u\rangle$ decreases so that m increases. In case the Rossby wave approaches a line where $\langle u\rangle = c$ the vertical wavenumber will grow without bounds. Typically the corresponding strong gradients in the wave field lead to nonlinear dissipation. A critical line will tend to act as a wave sink. Likewise there is the analogous case of downward Rossby-wave propagation into regions with decreasing wind, i.e., $\partial\langle u\rangle/\partial z > 0$, so that again a critical line with $\langle u\rangle = c$ can be approached.

Consequences for the Propagation of Rossby Waves into the Stratospere

A critical reader might have realized that the basic assumptions of WKB theory do not hold anymore when Rossby waves are reflected, since then the vertical wavelength diverges. Close to critical lines care is at place as well. Nonetheless, WKB keeps its validity at least qualitatively. We therefore use it here and note first that synoptic-scale Rossby waves, e.g., due to baroclinic instability, are hardly able to propagate into the stratosphere. This is because u_c becomes very small if the zonal wavenumber k is large. Take an example: In midlatitudes the meridional gradient of potential vorticity is approximated well by that of planetary vorticity, i.e., $\partial\langle\pi\rangle/\partial y \approx \beta = (2\Omega/a)\cos\phi_0 \approx 1.6 \cdot 10^{-11}\mathrm{m}^{-1}\mathrm{s}^{-1}$. Assuming $L_{di} \approx 1000\,\mathrm{km}$, one obtains with this, a typical synoptic-scale zonal wavenumber $k = 2\pi/(1000\,\mathrm{km})$ and $l = 0$ the estimate $u_c \approx 0.4\,\mathrm{m/s}$. Hence a synoptic-scale Rossby wave, initially in the extremely narrow frequency band $\langle u\rangle - u_c < c < \langle u\rangle$, will most inevitably approach in its vertical propagation reflection points and critical lines.

The situation differs for planetary Rossby waves. $k = 2\pi/(10000\,\mathrm{km})$, e.g., leads to $u_c = 25\,\mathrm{m/s}$, and in the limit $k \to 0$ one obtains $u_c \to 64\,\mathrm{m/s}$. Hence, the frequency band allowed for long planetary waves is much broader so that their vertical propagation into the stratosphere is possible. This is the case especially in winter, while in summer planetary Rossby waves will typically be absorbed at critical lines. Most of these waves are stationary, with $c \approx 0$, because they are typically due to the land–sea contrast or forced by continental orography. Whenever they approach a critical line where $\langle u\rangle \approx 0$, they will be absorbed. In summer this is nearly unavoidable, because the zonal-mean zonal wind changes its direction in the lower stratosphere (Fig. 8.2). The seasonal dependence of planetary waves in the stratosphere is illustrated in Fig. 8.3, where the daily-mean eddy geopotential is shown on the 1-mb level for a typical northern-hemisphere winter and summer day each. Clearly there are virtually no planetary waves on the respective summer side of the stratosphere.

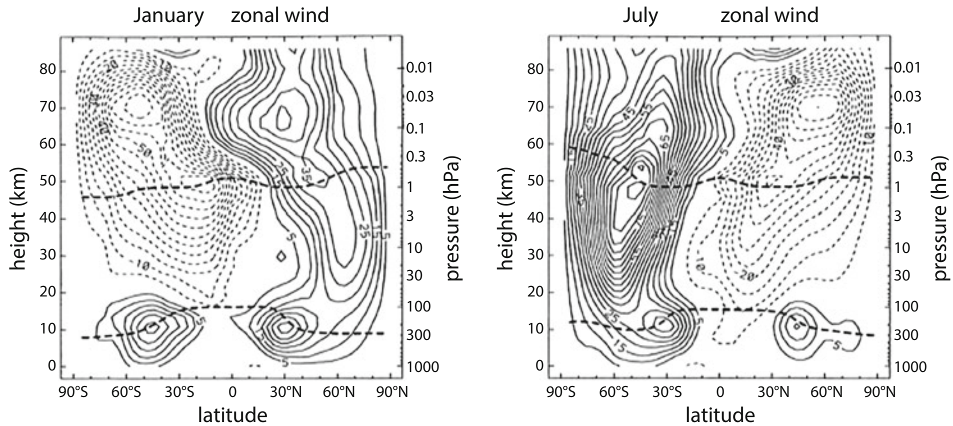

Fig. 8.2 Zonal-wind climatology in the lower and middle atmosphere (Randel et al. 2004).

8.2.3 Summary

The propagation of Rossby waves through a spatially and time-dependent zonal-mean atmosphere is captured well, at least on the qualitative level, by *WKB theory.*

- It assumes that the *spatial and time scales of variability of the atmosphere are large in comparison with the corresponding eddy wavelengths and periods.* The same assumption is made on the scales of variation of frequency and wavenumber themselves. Moreover, we assume *small wave amplitudes.* This leads to a general *dispersion relation* that the waves must fulfill at all locations and times. From this follow the *ray equations* that predict the variation of wavenumbers and frequency along rays parallel to the instantaneous group velocity. In a zonal-mean atmosphere the zonal wavenumber is invariant. In case the atmosphere is time-independent, frequency does not change either.
- Vertical wave propagation is influenced considerably by *critical lines and reflection lines.* The vertical wavenumber diverges at critical lines, and waves typically dissipate there by wave breaking. Reflection lines lead to a reversion of the vertical propagation direction. Similar behavior can also be found for meridional propagation. Wave reflection and critical lines explain why *synoptic-scale waves are hardly able to propagate from the troposphere into the stratosphere,* and why *planetary waves do so typically only in winter.*

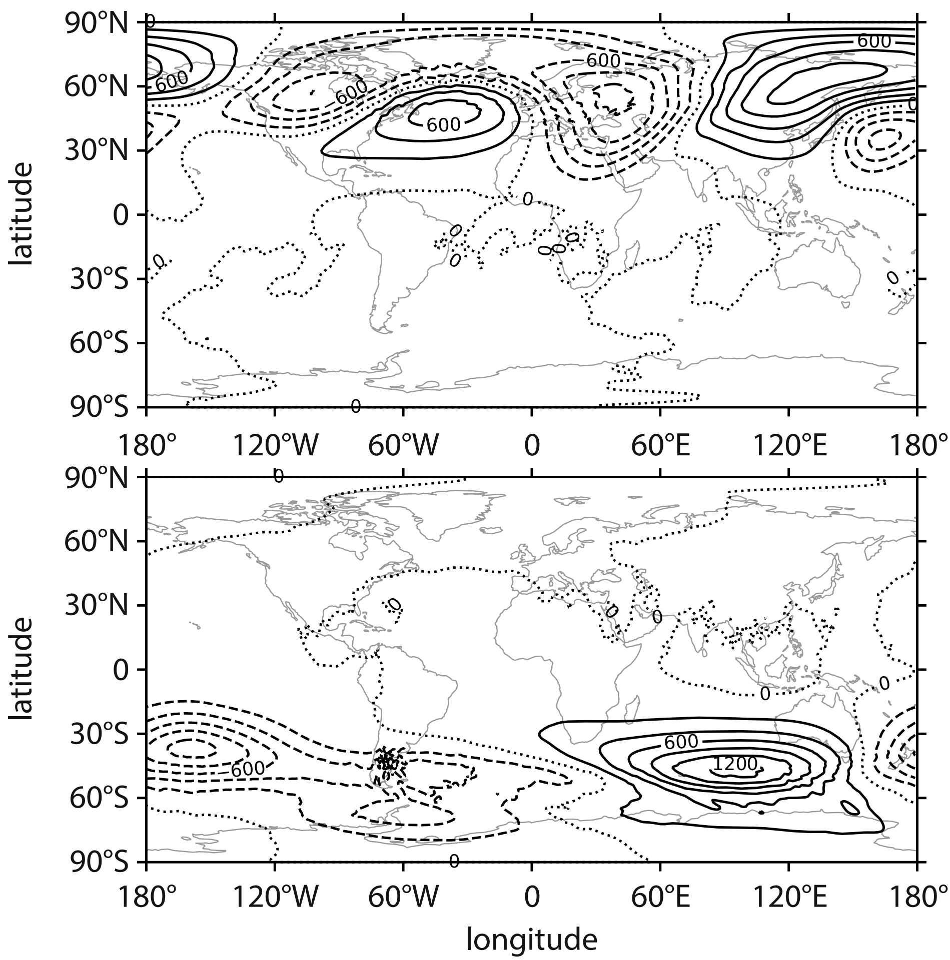

Fig. 8.3 Daily average, on the 1-mb pressure level in ERA5 data (Hersbach et al. 2020), of the eddy geopotential height (contour interval 200m, dashed lines indicating negative values) for January 1, 2010 (top panel) and July 1, 2010 (bottom).

8.3 The Eliassen–Palm Flux

The key for predicting the wave amplitudes is the Eliassen–Palm flux. We first derive it, pointing out its relation to the meridional flux of the potential vorticity. Finally its relevance for WKB theory wave amplitudes will be discussed.

8.3.1 Definition

Initially we want to take a look at the wave-induced meridional flux of the potential vorticity. Without zonal averaging the flux can be written, using (8.42) and (8.12),

$$v'_g\pi' = v'_g\zeta' + \frac{v'_g}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\frac{f_0}{N^2}b'\right) \tag{8.103}$$

Herein one has

$$\frac{v'_g}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\frac{f_0}{N^2}b'\right) = \frac{f_0}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\frac{v'_g b'}{N^2}\right) - f_0\frac{\partial v'_g}{\partial z}\frac{b'}{N^2} \tag{8.104}$$

Because of (8.6) and (8.9) the thermal-wind relation for the meridional wind is

$$f_0\frac{\partial v'_g}{\partial z} = \frac{\partial b'}{\partial x} \tag{8.105}$$

so that

$$\frac{v'_g}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\frac{f_0}{N^2}b'\right) = \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\, f_0\frac{v'_g b'}{N^2}\right) - \frac{\partial}{\partial x}\left(\frac{b'^2}{2\,N^2}\right) \tag{8.106}$$

From the definition (8.12) of the relative vorticity we obtain

$$v'_g\zeta' = v'_g\left(\frac{\partial v'_g}{\partial x} - \frac{\partial u'_g}{\partial y}\right) = -\frac{\partial}{\partial y}\left(u'_g v'_g\right) + u'_g\frac{\partial v'_g}{\partial y} + \frac{\partial}{\partial x}\frac{{v'_g}^2}{2} \tag{8.107}$$

Moreover, because the geostrophic wind has no divergence,

$$\frac{\partial u'_g}{\partial x} + \frac{\partial v'_g}{\partial y} = 0 \tag{8.108}$$

one has

$$u'_g\frac{\partial v'_g}{\partial y} = -\frac{\partial}{\partial x}\frac{u'^2_g}{2} \tag{8.109}$$

and therefore

$$v'_g\zeta' = -\frac{\partial}{\partial y}\left(u'_g v'_g\right) + \frac{\partial}{\partial x}\frac{{v'_g}^2 - {u'_g}^2}{2} \tag{8.110}$$

so that

$$\boxed{v'_g\pi' = \frac{\partial}{\partial x}\left(\frac{{v'_g}^2 - {u'_g}^2}{2} - \frac{1}{2}\frac{b'^2}{N^2}\right) - \frac{\partial}{\partial y}\left(u'_g v'_g\right) + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\, f_0\frac{v'_g b'}{N^2}\right)} \tag{8.111}$$

The zonal mean gives

$$\langle v_g'\pi'\rangle = \frac{1}{\overline{\rho}}\nabla\cdot\boldsymbol{\mathcal{F}} \tag{8.112}$$

where

$$\boldsymbol{\mathcal{F}} = -\overline{\rho}\langle u_g'v_g'\rangle\mathbf{e}_y + \frac{\overline{\rho}f_0}{N^2}\langle v_g'b'\rangle\mathbf{e}_z \tag{8.113}$$

is the Eliassen–Palm flux, with its divergence the meridional flux of the potential vorticity. The meridional component of the Eliassen–Palm flux is the negative wave-induced meridional flux of zonal momentum, and its vertical component is obtained from the meridional flux of buoyancy or potential temperature.

8.3.2 The Eliassen–Palm Relationship

The relationship between the meridional flux of potential vorticity and the Eliassen–Palm flux derived above can be used to rewrite the enstrophy Eq. (8.37) in order to facilitate the prediction of wave amplitudes. For this purpose we assume that the *meridional gradient of the mean potential vorticity only varies on time scales that are larger than the ones corresponding to the wave amplitudes,* i.e.,

$$\left|\frac{\partial}{\partial t}\frac{\partial\langle\pi\rangle}{\partial y}\bigg/\frac{\partial\langle\pi\rangle}{\partial y}\right| \ll \left|\frac{\partial}{\partial t}\frac{\langle\pi'^2\rangle}{2}\bigg/\frac{\langle\pi'^2\rangle}{2}\right| \tag{8.114}$$

which is not a bad assumption as long as the meridional gradient of the mean potential vorticity is dominated by the steady β-term. Dividing (8.37) by $\partial\langle\pi\rangle/\partial y$, and taking (8.114) and (8.112) into account, one obtains

$$\frac{\partial\mathcal{A}}{\partial t} + \nabla\cdot\boldsymbol{\mathcal{F}} = \mathcal{D} \tag{8.115}$$

with

$$\mathcal{A} = \frac{\overline{\rho}\langle\pi'^2/2\rangle}{\partial\langle\pi\rangle/\partial y} \tag{8.116}$$

the *wave-action density* and

$$\mathcal{D} = \frac{\overline{\rho}\langle D'\pi'\rangle}{\partial\langle\pi\rangle/\partial y} \tag{8.117}$$

the corresponding non-conservative sink or source. The integral of wave-action density, wave action, is conserved, i.e.,

$$\frac{d}{dt}\int dydz\mathcal{A} = 0 \tag{8.118}$$

if $\mathcal{D} = 0$ and the normal components of the Eliassen–Palm flux vanish at the boundaries of the integration domain. This is fulfilled for the case of the β-channel, where

$$y = 0, L_y : \qquad \overline{\rho}\langle u_g' v_g' \rangle = 0 \tag{8.119}$$

$$z = 0, \infty : \qquad \frac{\overline{\rho} f_0}{N^2} \langle v_g' b' \rangle = 0 \tag{8.120}$$

The meridional boundary conditions follow from the zero normal flow there, the upper boundary condition from the zero limit of the reference-atmosphere density, and the lower boundary condition from the zonal mean of (8.21) without friction and heating, if $\partial \langle b \rangle / \partial t|_{z=0} = 0$. Note that there is no similar conservation law for the wave part of energy and enstrophy, i.e.,

$$\frac{d}{dt} \int dydz \frac{1}{2} \left\langle \left| \nabla_h \psi' \right|^2 + \frac{b'^2}{N^2} \right\rangle \neq 0 \tag{8.121}$$

$$\frac{d}{dt} \int dydz \frac{1}{2} \left\langle \pi'^2 \right\rangle \neq 0 \tag{8.122}$$

not even in the conservative case! This is due to the fact that waves and mean flow can exchange energy and enstrophy. The influence the mean flow has on the wave enstrophy can be read directly from the enstrophy equation. An exchange of energy occurs, e.g., in the context of a baroclinic instability.

8.3.3 Wave Action and Eliassen–Palm Flux Within WKB Theory

The relation between the wave-action density and the Eliassen–Palm flux requires small wave amplitudes. If one also assumes scale separation in the spirit of WKB, further simplifications emerge that shed more light on the meaning of both quantities.

We begin with the representation (8.93) of the wave streamfunction in the context of WKB theory. Because every zonal wavenumber

$$k = \frac{\partial \alpha_k}{\partial x} \tag{8.123}$$

is invariant in space and time, one can write the phase

$$\alpha_k(\mathbf{x}, t) = kx + \beta_k(y, z, t) \tag{8.124}$$

Then the streamfunction becomes

$$\psi' = \frac{1}{\sqrt{\overline{\rho}}} \sum_k A_k e^{i(kx + \beta_k)} \tag{8.125}$$

It is real-valued so that

$$\frac{1}{\sqrt{\bar{\rho}}}\sum_k A_k e^{i(kx+\beta_k)} = \psi' = \psi'^* = \frac{1}{\sqrt{\bar{\rho}}}\sum_k A_k^* e^{-i(kx+\beta_k)} = \frac{1}{\sqrt{\bar{\rho}}}\sum_k A_{-k}^* e^{i(kx-\beta_{-k})} \tag{8.126}$$

Hence we obtain the identities

$$A_{-k} = A_k^* \tag{8.127}$$

$$\beta_{-k} = -\beta_k \tag{8.128}$$

and furthermore

$$l_{-k} = \frac{\partial \beta_{-k}}{\partial y} = -\frac{\partial \beta_k}{\partial y} = -l_k \tag{8.129}$$

and likewise

$$m_{-k} = -m_k \tag{8.130}$$

$$\omega_{-k} = -\omega_k \tag{8.131}$$

These results shall be used in reformulating wave-action density and Eliassen–Palm flux.

From the relations (8.65) and (8.80), the latter holding separately for each k, one can see that the potential vorticity of the waves is

$$\pi' = -\frac{1}{\sqrt{\bar{\rho}}}\sum_k \pi_k e^{i(kx+\beta_k)} \tag{8.132}$$

with

$$\pi_k = \left[\left(k^2 + l_k^2\right) + \frac{f_0^2}{N^2} m_k^2 + \frac{1}{4L_{di}^2}\right] A_k \tag{8.133}$$

so that

$$\langle \pi'^2 \rangle = \frac{1}{\bar{\rho}} \sum_k \sum_{k'} \left\langle \pi_k e^{i(kx+\beta_k)} \pi_{k'} e^{i(k'x+\beta_{k'})} \right\rangle \tag{8.134}$$

For reasons of simplicity[2] we now assume that the amplitudes A_k do not depend on x. One obtains

$$\langle \pi'^2 \rangle = \frac{1}{\bar{\rho}} \sum_k \sum_{k'} \pi_k e^{i\beta_k} \pi_{k'} e^{i\beta_{k'}} \left\langle e^{i(k+k')x} \right\rangle \tag{8.135}$$

But due to (8.94) one has

$$\left\langle e^{i(k+k')x} \right\rangle = \delta_{k',-k} \tag{8.136}$$

so that

$$\langle \pi'^2 \rangle = \frac{1}{\bar{\rho}} \sum_k \pi_k e^{i\beta_k} \pi_{-k} e^{i\beta_{-k}} \tag{8.137}$$

[2] A corresponding generalization is possible.

Using (8.127)–(8.130), which can also be written

$$\pi_{-k} = \pi_k^* \tag{8.138}$$

one obtains

$$\langle \pi'^2 \rangle = \frac{1}{\overline{\rho}} \sum_k |\pi_k|^2 = \frac{1}{\overline{\rho}} \sum_k \left(k^2 + l_k^2 + \frac{f_0^2}{N^2} m_k^2 + \frac{1}{4L_{di}^2} \right)^2 |A_k|^2 \tag{8.139}$$

Inserting this into the definition (8.116) yields

$$\mathcal{A} = \sum_k \mathcal{A}_k \tag{8.140}$$

$$\mathcal{A}_k = \frac{1}{2\partial\langle\pi\rangle/\partial y} \left(k^2 + l_k^2 + \frac{f_0^2}{N^2} m^2 + \frac{1}{4L_{di}^2} \right)^2 |A_k|^2 \tag{8.141}$$

That is, the wave-action density can be written as a sum of contributions from the individual WKB solutions.

It can be expected that the corresponding prognostic Eq. (8.115) could be used for predicting the wave amplitudes. In order to do so the Eliassen–Palm flux has to be rewritten first. We begin with the meridional component. For the meridional momentum flux used in this expression the components of the geostrophic wind are required. Using (8.125) those are, neglecting all slow variations of the amplitude,

$$u'_g = -\frac{\partial \psi'}{\partial y} = -\frac{1}{\sqrt{\overline{\rho}}} \sum_k A_k i l_k e^{i(kx+\beta_k)} \tag{8.142}$$

$$v'_g = \frac{\partial \psi'}{\partial x} = \frac{1}{\sqrt{\overline{\rho}}} \sum_k A_k i k e^{i(kx+\beta_k)} \tag{8.143}$$

Analogous to the calculation for (8.139) one obtains

$$\langle u'_g v'_g \rangle = -\frac{1}{\overline{\rho}} \sum_k |A_k|^2 \, k l_k \tag{8.144}$$

and thus, using (8.113),

$$\mathcal{F}_y = \sum_k |A_k|^2 \, k l_k \tag{8.145}$$

It is now important that the meridional component of the group velocity is, using (8.85) and (8.88),

$$c_{gy,k} = \frac{\partial \omega_k}{\partial l} = \frac{2lk\partial\langle\pi\rangle/\partial y}{\left(k^2 + l^2 + \frac{f_0^2}{N^2} m^2 + \frac{1}{4L_{di}^2} \right)^2} \tag{8.146}$$

Comparing this with (8.140) one finally obtains

$$\mathcal{F}_y = \sum_k \mathcal{F}_{y,k} \tag{8.147}$$

with

$$\mathcal{F}_{k,y} = c_{gy,k}\mathcal{A}_k \tag{8.148}$$

That is, the meridional Eliassen–Palm flux is the sum of the meridional fluxes of the wave-action density of the contributing WKB components. These are given by the product of the meridional group velocity with the wave-action density.

A similar result can be obtained for the vertical flux. Here we need the meridional buoyancy flux, and there the buoyancy fluctuations. Due to (8.9) those are approximately

$$b' = f_0 \frac{\partial \psi'}{\partial z} = \frac{i f_0}{\sqrt{\overline{\rho}}} \sum_k A_k m_k e^{i(kx+\beta_k)} \tag{8.149}$$

Together with (8.143) this leads, in a manner similar to the derivation of (8.139), to

$$\langle v_g' b' \rangle = \frac{f_0}{\overline{\rho}} \sum_k |A_k|^2 \, k m_k \tag{8.150}$$

With (8.113) one obtains

$$\mathcal{F}_z = \frac{f_0^2}{N^2} \sum_k k m_k \, |A_k|^2 \tag{8.151}$$

Moreover, due to (8.85) and (8.88), the vertical component of the group velocity is

$$c_{gz,k} = \frac{\partial \omega_k}{\partial m} = \frac{f_0^2}{N^2} \frac{2km\partial\langle\pi\rangle/\partial y}{\left(k^2 + l_k^2 + \frac{f_0^2}{N^2}m^2 + \frac{1}{4L_{di}^2}\right)^2} \tag{8.152}$$

so that

$$\mathcal{F}_z = \sum_k \mathcal{F}_{k,z} \tag{8.153}$$

with

$$\mathcal{F}_{k,z} = c_{gz,k}\mathcal{A}_k \tag{8.154}$$

Hence also the vertical component of the Eliassen–Palm flux is the sum of the corresponding fluxes of the wave-action density of the contributing WKB components, which themselves are a product of meridional group velocity and wave-action density. Wave-action density is transported with its group velocity, and the corresponding flux is the Eliassen–Palm flux. In summary, one finds within WKB theory

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}} = \mathcal{D} \qquad \mathcal{A} = \sum_k \mathcal{A}_k \qquad \boldsymbol{\mathcal{F}} = \sum_k \boldsymbol{\mathcal{F}}_k \tag{8.155}$$

$$\mathcal{A}_k = \frac{1}{2\partial\langle\pi\rangle/\partial y} \left(k^2 + l_k^2 + \frac{f_0^2}{N^2} m^2 + \frac{1}{4L_{di}^2} \right)^2 |A_k|^2 \tag{8.156}$$

$$\boldsymbol{\mathcal{F}}_k = \sum_k \mathbf{c}_{g,k} \mathcal{A}_k \tag{8.157}$$

It should be stressed that, due to the linearity of the dynamics, the individual WKB components are not coupled. Thus it is possible to predict the amplitude of every individual component independently from the amplitudes of the other components, by considering only the desired component in the equations above; i.e., one solves

$$\frac{\partial \mathcal{A}_k}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{F}}_k = \mathcal{D}_k \tag{8.158}$$

where $\mathcal{D}_k$ is the contribution of the zonal wavenumber k to $\mathcal{D}$.

8.3.4 Summary

The key to an understanding of the development of the Rossby-wave amplitudes, within linear theory, is given by the *Eliassen–Palm flux.*

- The *meridional flux of potential vorticity agrees with the Eliassen–Palm flux divergence.*
- *In the conservative case, a change in the wave-action density is due to Eliassen–Palm flux divergence or convergence.* Hence its volume integral is conserved, under regular conditions. There are no such conservation theorems for wave energy and wave enstrophy, because waves and mean flow exchange energy and enstrophy.
- Within WKB theory the *Eliassen–Palm flux* is identical with the *product of group velocity and wave-action density.*

8.4 The Transformed Eulerian Mean (TEM)

Up to here the focus has been on wave dynamics and the question how waves are influenced by the mean flow. Now we reverse the perspective and address the question how *waves influence the mean flow.* In addition, the next sections also discuss the combined impact of mean flow and waves on the average *mass transport.*

8.4.1 The TEM in the Context of Quasigeostrophy

Although the matter could be treated in a more general manner, we here focus on synoptic-scale dynamics on the β-plane. The goal is to obtain equations for the zonal-mean flow. The zonally averaged streamfunction has gradients in meridional as well as in vertical directions. These two gradients, which are the zonal mean of the geostrophic zonal wind and buoyancy, are to be studied in terms of their evolution in time. Beyond this we are also interested in the zonal-mean circulation via $\langle v \rangle$ and $\langle w \rangle$.

To begin with, we recall that the continuity equation is to leading orders

$$\nabla \cdot (\overline{\rho} \mathbf{v}) = 0 \tag{8.159}$$

as can be seen, e.g., from the contributions of $\mathcal{O}(1)$ and $\mathcal{O}(Ro)$ in (6.58). Its zonal mean then is

$$\boxed{\nabla \cdot (\overline{\rho} \langle \mathbf{v} \rangle) = 0} \tag{8.160}$$

where both $\langle v \rangle$ and $\langle w \rangle$ are ageostrophic to leading order! Subtracting this from (8.159) yields

$$\nabla \cdot \left(\overline{\rho} \mathbf{v}' \right) = 0 \tag{8.161}$$

Next we consider the zonal-momentum equation

$$\frac{\partial u}{\partial t} + (\mathbf{v} \cdot \nabla) u - f v = -\frac{1}{\overline{\rho}} \frac{\partial p}{\partial x} + F \tag{8.162}$$

where F comprises all non-conservative forces, in particular friction. The corresponding zonal mean is

$$\frac{\partial \langle u \rangle}{\partial t} + \langle (\mathbf{v} \cdot \nabla) u \rangle - f \langle v \rangle = \langle F \rangle \tag{8.163}$$

with

$$\langle (\mathbf{v} \cdot \nabla) u \rangle = (\langle \mathbf{v} \rangle \cdot \nabla) \langle u \rangle + \left\langle \left(\mathbf{v}' \cdot \nabla \right) u' \right\rangle \tag{8.164}$$

Here the mean-flow contribution is

$$(\langle \mathbf{v} \rangle \cdot \nabla) \langle u \rangle = \langle v \rangle \frac{\partial \langle u \rangle}{\partial y} + \langle w \rangle \frac{\partial \langle u \rangle}{\partial z} \tag{8.165}$$

Using the zonal mean of the relative vorticity

$$\langle \zeta \rangle = \left\langle \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right\rangle = -\frac{\partial \langle u \rangle}{\partial y} \tag{8.166}$$

one can rewrite this as

$$(\langle \mathbf{v} \rangle \cdot \nabla) \langle u \rangle = -\langle \zeta \rangle \langle v \rangle + \langle w \rangle \frac{\partial \langle u \rangle}{\partial z} \tag{8.167}$$

The eddy part is, using (8.161),

$$\left\langle\left(\mathbf{v}' \cdot \nabla\right) u'\right\rangle = \frac{1}{\overline{\rho}}\left\langle\left(\overline{\rho}\mathbf{v}' \cdot \nabla\right) u'\right\rangle = \frac{1}{\overline{\rho}}\nabla \cdot \left\langle\overline{\rho}\mathbf{v}'u'\right\rangle = \frac{\partial}{\partial y}\left\langle v'u'\right\rangle + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left\langle\overline{\rho}w'u'\right\rangle \tag{8.168}$$

Inserting (8.167) and (8.168) into (8.165) yields

$$\langle(\mathbf{v} \cdot \nabla) u\rangle = -\langle\zeta\rangle\langle v\rangle + \langle w\rangle\frac{\partial\langle u\rangle}{\partial z} + \frac{\partial}{\partial y}\left\langle v'u'\right\rangle + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left\langle\overline{\rho}w'u'\right\rangle \tag{8.169}$$

Thus the zonal mean of the zonal momentum equations becomes

$$\frac{\partial\langle u\rangle}{\partial t} - (\langle\zeta\rangle + f)\langle v\rangle + \langle w\rangle\frac{\partial\langle u\rangle}{\partial z} = -\frac{\partial}{\partial y}\langle u'v'\rangle - \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\langle\overline{\rho}u'w'\rangle + \langle F\rangle \tag{8.170}$$

In synoptic scaling not all terms are equally large. In order to see this, we recall that the synoptic-scale horizontal wind can be expanded

$$u = U\left(\hat{u}_0 + Ro\,\hat{u}_1 + O\left(Ro^2\right)\right) \tag{8.171}$$
$$v = U\left(\hat{v}_0 + Ro\,\hat{v}_1 + O\left(Ro^2\right)\right) \tag{8.172}$$

where as before U is a typical horizontal-wind scale, and $Ro = U/f_0L \ll 1$ the Rossby number, with L the horizontal length scale. $\hat{u}_i$ and $\hat{v}_i$ have no dimension and are all $\mathcal{O}(1)$. The non-dimensional geostrophic meridional wind can be calculated from the non-dimensional streamfunction $\hat{\psi}$ via

$$\hat{v}_0 = \frac{\partial\hat{\psi}}{\partial\hat{x}} \tag{8.173}$$

with $\hat{x} = x/L$ being the non-dimensional coordinate in zonal direction. Therefore one has

$$\langle\hat{v}_0\rangle = 0 \tag{8.174}$$

and thus

$$\langle u\rangle = \mathcal{O}(U) \tag{8.175}$$
$$\langle v\rangle = \mathcal{O}(Ro\,U) \tag{8.176}$$

Moreover, we have seen in the derivation of quasigeostrophic theory that the vertical wind is

$$\langle w\rangle = \mathcal{O}\left(Ro\frac{H}{L}U\right) \tag{8.177}$$

where H is the synoptic vertical length scale. For the planetary vorticity we have derived from $L/a = \mathcal{O}(Ro)$ that

$$f = f_0\left[1 + \mathcal{O}(Ro)\right] \tag{8.178}$$

while for the relative vorticity one has

$$\langle\zeta\rangle = \mathcal{O}\left(\frac{U}{L}\right) \tag{8.179}$$

Invoking as time scale the advective time scale $T = L/U$, one hence obtains the following for the different terms in the zonal-mean zonal-momentum equation

$$\frac{\partial\langle u\rangle}{\partial t} = \mathcal{O}\left(\frac{U^2}{L}\right) \tag{8.180}$$

$$(\langle\zeta\rangle + f)\,\langle v\rangle = f_0\langle v\rangle + \mathcal{O}\left(Ro\frac{U^2}{L}\right) \tag{8.181}$$

$$f_0\langle v\rangle = \mathcal{O}\left(\frac{U^2}{L}\right) \tag{8.182}$$

$$\langle w\rangle\frac{\partial\langle u\rangle}{\partial z} = \mathcal{O}\left(Ro\frac{U^2}{L}\right) \tag{8.183}$$

$$\frac{\partial}{\partial y}\langle u'v'\rangle = \mathcal{O}\left(\frac{U^2}{L}\right) \tag{8.184}$$

$$\frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\langle\overline{\rho}u'w'\rangle = \mathcal{O}\left(Ro\frac{U^2}{L}\right) \tag{8.185}$$

One therefore finds in good approximation

$$\boxed{\frac{\partial\langle u\rangle}{\partial t} - f_0\langle v\rangle = -\frac{\partial}{\partial y}\langle u'v'\rangle + \langle F\rangle} \tag{8.186}$$

On the other hand the entropy equation

$$\frac{D\theta}{Dt} = \frac{q\theta}{c_pT} \tag{8.187}$$

can be rewritten, using the decomposition

$$\theta = \overline{\theta}(z) + \tilde{\theta}(x, y, z, t) \tag{8.188}$$

as a buoyancy equation

$$\frac{Db}{Dt} + N^2 w\left(1 + \frac{\tilde{\theta}}{\overline{\theta}}\right) = Q \tag{8.189}$$

with

$$Q = \frac{gq\theta}{c_pT\overline{\theta}} \tag{8.190}$$

In synoptic scaling

$$\frac{\tilde{\theta}}{\overline{\theta}} = \mathcal{O}(Ro^2) \tag{8.191}$$

so that the buoyancy equation becomes in good approximation

$$\frac{Db}{Dt} + N^2 w = Q \tag{8.192}$$

The corresponding zonal mean derived in the same way as the zonal-mean zonal-momentum equation is

$$\frac{\partial \langle b \rangle}{\partial t} + \langle v \rangle \frac{\partial \langle b \rangle}{\partial y} + \left(N^2 + \frac{\partial \langle b \rangle}{\partial z} \right) \langle w \rangle = -\frac{\partial}{\partial y} \langle v' b' \rangle - \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \langle \overline{\rho} w' b' \rangle + \langle Q \rangle \tag{8.193}$$

Again, not all terms are in synoptic scaling of the same order of magnitude. To see this we note that from (8.191)

$$b = g\,\mathcal{O}(Ro^2) \tag{8.194}$$

Furthermore, in the derivation of quasigeostrophic theory we have assumed

$$\frac{1}{\overline{\theta}} \frac{d\overline{\theta}}{d\hat{z}} = \mathcal{O}(Ro) \tag{8.195}$$

so that

$$N^2 = \frac{g}{\overline{\theta}} \frac{d\overline{\theta}}{dz} = \frac{g}{H} \frac{1}{\overline{\theta}} \frac{d\overline{\theta}}{d\hat{z}} = \frac{g}{H} \mathcal{O}(Ro) \tag{8.196}$$

This results in the following scale estimates in the zonally averaged buoyancy equation

$$\frac{\partial \langle b \rangle}{\partial t} = \mathcal{O}\left(Ro^2\, g \frac{U}{L} \right) \tag{8.197}$$

$$\langle v \rangle \frac{\partial \langle b \rangle}{\partial y} = \mathcal{O}\left(Ro^3\, g \frac{U}{L} \right) \tag{8.198}$$

$$N^2 \langle w \rangle = \mathcal{O}\left(Ro^2\, g \frac{U}{L} \right) \tag{8.199}$$

$$\langle w \rangle \frac{\partial \langle b \rangle}{\partial z} = \mathcal{O}\left(Ro^3\, g \frac{U}{L} \right) \tag{8.200}$$

$$\frac{\partial}{\partial y} \langle v' b' \rangle = \mathcal{O}\left(Ro^2\, g \frac{U}{L} \right) \tag{8.201}$$

$$\frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \langle \overline{\rho} w' b' \rangle = \mathcal{O}\left(Ro^3\, g \frac{U}{L} \right) \tag{8.202}$$

Hence we can write in good approximation

$$\boxed{\frac{\partial \langle b \rangle}{\partial t} + N^2 \langle w \rangle = -\frac{\partial}{\partial y} \langle v' b' \rangle + \langle Q \rangle} \tag{8.203}$$

It is important to note that the two zonal-mean Eqs. (8.186) and (8.203) are not independent from each other, because the zonal averages of the zonal wind and buoyancy are linked by the thermal-wind relation

$$\frac{\partial \langle u \rangle}{\partial z} = -\frac{1}{f_0} \frac{\partial \langle b \rangle}{\partial y} \tag{8.204}$$

Provided the wave fluxes, heating, and friction are given, (8.160), (8.186), (8.203), and (8.204) determine all three components of $\langle \mathbf{v} \rangle$ and $\langle b \rangle$.

Moreover, the mean continuity Eq. (8.160) combined with the Helmholtz theorem implies that there is a mass streamfunction ψ_m so that

$$\langle v \rangle = -\frac{1}{\overline{\rho}} \frac{\partial \psi_m}{\partial z} \tag{8.205}$$

$$\langle w \rangle = \frac{1}{\overline{\rho}} \frac{\partial \psi_m}{\partial y} \tag{8.206}$$

The product of the mean flow and density, the mean mass flux, follows the isolines of the mass streamfunction in a way, such that the flow circulates clockwise around a maximum and counterclockwise around a minimum. A typical example is depicted in Fig. 8.4.

The direct zonal mean of meridional and vertical wind is the so-called *Eulerian mean*. For various reasons, applying this type of averaging has proven to be impractical:

- Steady solutions of the mean buoyancy Eq. (8.203) imply, e.g.,

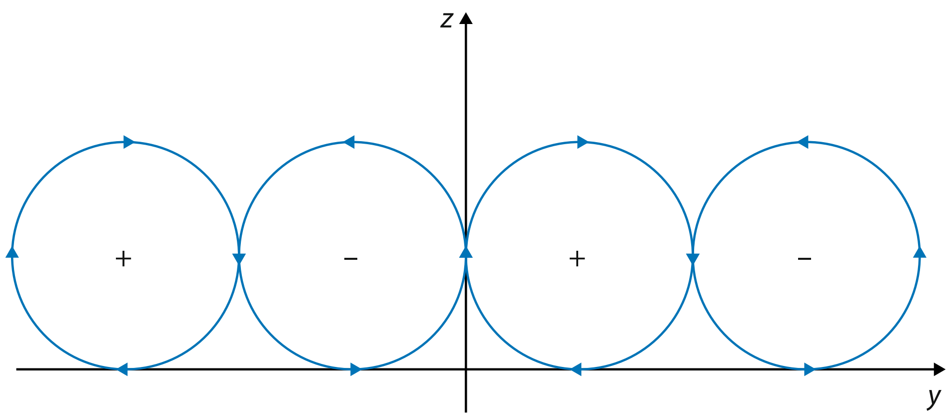

Fig. 8.4 An example for the relation between the mass streamfunction and the zonal-mean circulation. The mean flow follows the isolines of the mass streamfunction in such a way that it circulates clockwise around a maximum and counterclockwise around a minimum.

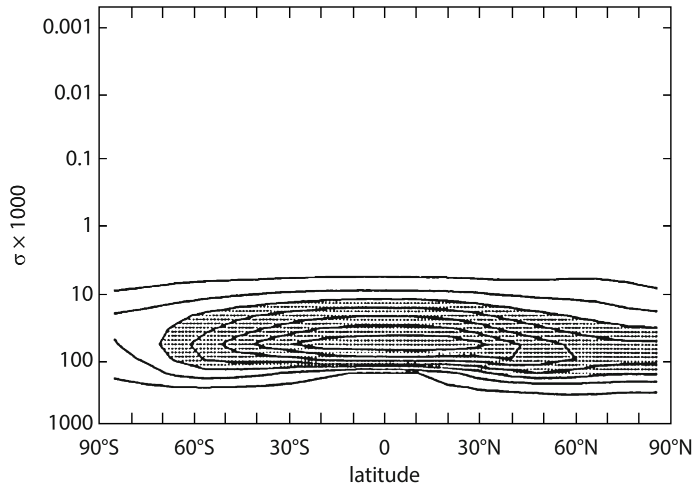

Fig. 8.5 Latitude–Altitude distribution of the photochemical ozone source in spring. Reprinted from (1994) with permission of Cambridge University Press.

$$\langle w \rangle = -\frac{1}{N^2}\left(\frac{\partial}{\partial y}\langle v'b'\rangle - \langle Q\rangle\right) \tag{8.207}$$

so that even without heating one obtains rising and sinking air masses. This motion is driven only by waves. In many cases it is more useful to express the averaged dynamics in a way so that the wave-driven part does not appear explicitly.

- Furthermore, the mean meridional and vertical mass transport, for example, of arbitrary tracers, is not represented by the Eulerian mean of the meridional circulation. As an example we show in Figs. 8.5 and 8.6 for stratospheric ozone its source distribution and the climatological ozone distribution itself, while Fig. 8.7 shows the Eulerian mean of the meridional circulation. Notably the source is in the tropics, whereas the maximum density of ozone is to be found in the polar regions. The mean circulation from Fig. 8.7 cannot explain this discrepancy.

As a remedy for the first issue we introduce a *residual* streamfunction[3]

$$\boxed{\psi^* = \psi_m + \frac{\overline{\rho}}{N^2}\langle v'b'\rangle} \tag{8.208}$$

so that

$$N^2\langle w\rangle + \frac{\partial}{\partial y}\langle v'b'\rangle = N^2\frac{1}{\overline{\rho}}\frac{\partial\psi^*}{\partial y} \tag{8.209}$$

[3] Here the asterisk does *not* indicate the complex conjugate.

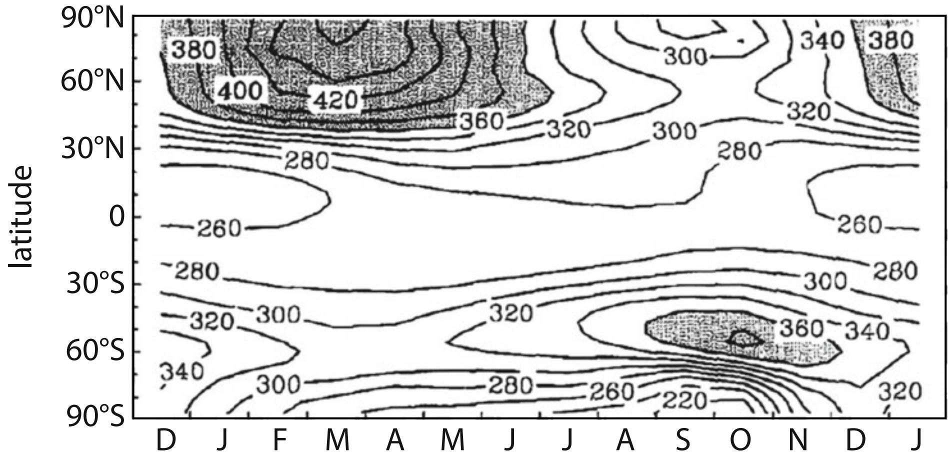

Fig. 8.6 Annual cycle of the latitudinal distribution of the vertically and zonally averaged ozone. Reprinted from (1995) with permission from Elsevier.

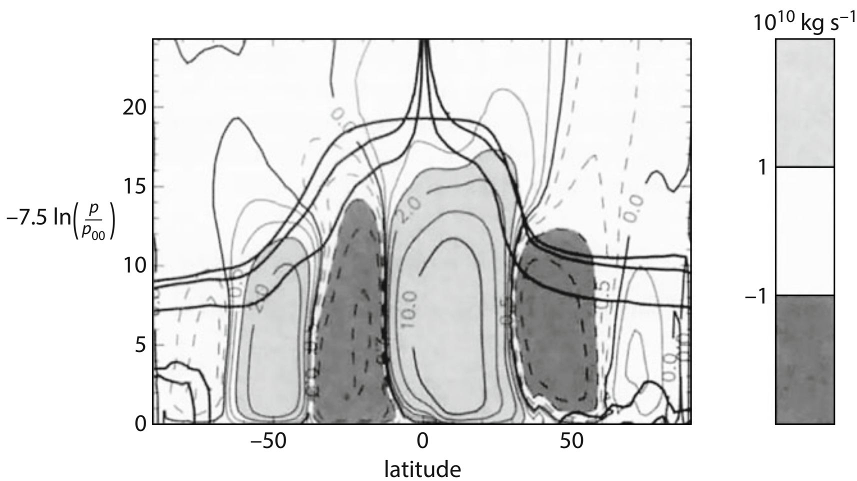

Fig. 8.7 Latitude–Altitude distribution of the mass streamfunction in the Eulerian mean for the winter of the northern hemisphere. One can clearly see the Hadley cells in the tropics and the Ferrel cells in mid-latitudes. The upper branch of the Ferrel cells is oriented from the poles to the equator, thus working against the transport of ozone from the equator to the poles. Reprinted from Juckes (2001).

The corresponding *residual-mean circulation* is then defined as

$$\langle v\rangle^* = -\frac{1}{\overline{\rho}}\frac{\partial\psi^*}{\partial z} = \langle v\rangle - \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left\langle\frac{\overline{\rho}}{N^2}v'b'\right\rangle \tag{8.210}$$

$$\langle w\rangle^* = \frac{1}{\overline{\rho}}\frac{\partial\psi^*}{\partial y} = \langle w\rangle + \frac{1}{N^2}\frac{\partial}{\partial y}\langle v'b'\rangle \tag{8.211}$$

for which

$$\nabla\cdot\left(\overline{\rho}\langle\mathbf{v}\rangle^*\right) = 0 \tag{8.212}$$

Substituting $\langle v\rangle$ in (8.186) one obtains the interesting result

$$\begin{aligned}\frac{\partial\langle u\rangle}{\partial t} - f_0\langle v\rangle^* &= -\frac{\partial}{\partial y}\langle u'v'\rangle + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left\langle f_0\frac{\overline{\rho}}{N^2}v'b'\right\rangle + \langle F\rangle \\ &= \frac{1}{\overline{\rho}}\nabla\cdot\boldsymbol{\mathcal{F}} + \langle F\rangle = \left\langle v'\pi'\right\rangle + \langle F\rangle\end{aligned} \tag{8.213}$$

In the transformed representation the Eliassen–Palm flux divergence is the wave forcing of the mean flow! Substituting as well the mean vertical wind in the buoyancy equation finishes the derivation of the mean equations of motion in the representation of the *transformed Eulerian mean (TEM).*

$$\frac{\partial\langle u\rangle}{\partial t} = f_0\langle v\rangle^* + \langle v'\pi'\rangle + \langle F\rangle \tag{8.214}$$

$$\frac{\partial\langle b\rangle}{\partial t} = -N^2\langle w\rangle^* + \langle Q\rangle \tag{8.215}$$

Given the wave fluxes, heating and friction, these determine together with (8.212) and (8.204) the residual-mean circulation in $\langle v\rangle^*$ and $\langle w\rangle^*$ and the zonal means of zonal wind and buoyancy. In this representation, a steady (residual) mean vertical wind is only due to the zonal-mean heating. What remains to be shown is that the residual-mean circulation represents the mean mass transport. For this purpose we will define in the following two chapters the mass-weighted mean circulation and then discuss its connection with the residual-mean circulation.

8.4.2 The Mass-Weighted Circulation in Isentropic Coordinates

The definition of the mass-weighted circulation is easier in isentropic coordinates. We recall that in these coordinates the vertical velocity

$$\dot{\theta} = \frac{q\theta}{c_p T} \tag{8.216}$$

is only induced diabatically. Because it vanishes in the absence of heating processes, including frictional heating and heat conduction, mass transport without heating in isentropic coordinates is purely horizontal. Therefore the continuity equation has the form

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot (\mathbf{u}\sigma) = H_\theta \tag{8.217}$$

in which the right-hand side

$$H_\theta = -\frac{\partial}{\partial \theta}\left(\sigma\dot{\theta}\right) \tag{8.218}$$

vanishes if there is no heating. The thickness

$$\sigma(x, y, \theta, t) = -\frac{1}{g}\frac{\partial p}{\partial \theta} \tag{8.219}$$

is a measure for the mass per area, at a given horizontal location and between two adjacent isentropic layers with potential temperature θ and $\theta + d\theta$. This can be understood by inspecting the hydrostatic relation

$$\partial p = -g\rho\partial z \tag{8.220}$$

whence

$$\sigma = \rho\frac{\partial z}{\partial \theta} \tag{8.221}$$

wherein $dm = \rho dz$ is the mass per area that hence can also be written

$$dm = \sigma d\theta \tag{8.222}$$

Now we consider the zonal mean of the continuity equation,

$$\frac{\partial\langle\sigma\rangle}{\partial t} + \frac{\partial}{\partial y}\langle v\sigma\rangle = \langle H_\theta\rangle \tag{8.223}$$

or

$$\frac{\partial\langle\sigma\rangle}{\partial t} + \frac{\partial}{\partial y}\left(\langle v\rangle\langle\sigma\rangle\right) = -\frac{\partial}{\partial y}\langle v'\sigma'\rangle + \langle H_\theta\rangle \tag{8.224}$$

If the Eulerian mean $\langle v\rangle$ of the meridional velocity were suitable for the description of the transport of the zonally averaged mass, the right-hand side would have not contributed in the absence of heating. This is obviously not the case, due to an eddy contribution termed *Stokes drift*. In the effective mass-weighted transport velocity this must be taken into account. The former is defined as

$$\boxed{\langle v\rangle_* = \frac{\langle v\sigma\rangle}{\langle\sigma\rangle} = \langle v\rangle + \frac{\langle v'\sigma'\rangle}{\langle\sigma\rangle}} \tag{8.225}$$

so that the mean continuity equation becomes

$$\boxed{\frac{\partial\langle\sigma\rangle}{\partial t}+\frac{\partial}{\partial y}\left(\langle v\rangle_*\langle\sigma\rangle\right)=\langle H_\theta\rangle} \tag{8.226}$$

Indeed $\langle v\rangle_*$ describes the effective mass transport. In good approximation one can also use it for the transport of arbitrary tracers.

In the steady case, e.g., when considering climatological means, the mass-weighted circulation can also be expressed via a mass streamfunction that is useful for diagnostic purposes. In this case we have, with $\langle\dot\theta\rangle_*=\langle\dot\theta\sigma\rangle/\langle\sigma\rangle$ the mass-weighted entropy velocity,

$$\frac{\partial}{\partial y}\left(\langle v\rangle_*\langle\sigma\rangle\right)+\frac{\partial}{\partial\theta}\left(\langle\dot\theta\rangle_*\langle\sigma\rangle\right)=0 \tag{8.227}$$

Hence there must be a mass streamfunction ψ_* so that

$$\langle v\rangle_*=-\frac{1}{\langle\sigma\rangle}\frac{\partial\psi_*}{\partial\theta} \tag{8.228}$$

$$\langle\dot\theta\rangle_*=\frac{1}{\langle\sigma\rangle}\frac{\partial\psi_*}{\partial y} \tag{8.229}$$

8.4.3 The Relation Between the Residual Circulation and the Mass-Weighted Circulation

The demonstration of the approximate identity between residual and mass-weighted circulation requires a transformation from the isentropic coordinates defining the mass-weighted circulation to the geometric z-coordinate that we have used for defining the residual circulation. It turns out to be helpful to first transform the mass-weighted circulation in an intermediate step to pressure coordinates and move from there to the geometric representation. For this purpose we first define for an arbitrary variable X, given in isentropic or pressure coordinates, the isentropic zonal mean

$$\langle X\rangle_\Theta(\theta)=\frac{1}{L_x}\int_0^{L_x}dx\,X\left(x,\theta\right) \tag{8.230}$$

and the isobaric mean

$$\langle X\rangle_P(p)=\frac{1}{L_x}\int_0^{L_x}dx\,X\left(x,p\right) \tag{8.231}$$

where we suppress for better readability the dependence on y and t, as will be the case throughout this chapter. The mass-weighted mean meridional velocity is in this notation

$$\langle v\rangle_*(\theta)=\langle v\rangle_\Theta(\theta)+\left[\frac{\langle v'\sigma'\rangle_\Theta}{\langle\sigma\rangle_\Theta}\right](\theta) \tag{8.232}$$

Let now $\langle p\rangle_\Theta(\theta)$ be the isentropic zonal-mean pressure so that the total pressure can be expressed in isentropic coordinates via

$$p(x,\theta) = \langle p\rangle_\Theta(\theta) + \eta\,(x,\theta) \tag{8.233}$$

Here η is the field of pressure deviations from the isentropic mean. Hence

$$\begin{aligned}\langle X\rangle_\Theta(\theta) &= \frac{1}{L_x}\int_0^{L_x} dx\, X\,[x,\, p\,(x,\theta)] = \frac{1}{L_x}\int_0^{L_x} dx\, X\,[x,\, \langle p\rangle_\Theta(\theta) + \eta\,(x,\theta)] \\ &\approx \frac{1}{L_x}\int_0^{L_x} dx\, \left\{X\,[x,\, \langle p\rangle_\Theta(\theta)] + \frac{\partial X}{\partial p}\,[x,\, \langle p\rangle_\Theta(\theta)]\,\eta\,(x,\theta)\right\}\end{aligned} \tag{8.234}$$

The first part of this integral already yields a mean in pressure coordinates. In order to achieve this also for the second part we need η in pressure coordinates as well; with this goal we first note that

$$\begin{aligned}\theta &= \theta\,[x,\, p(x,\theta)] = \theta\,[x,\, \langle p\rangle_\Theta(\theta) + \eta\,(x,\theta)] \\ &\approx \theta\,[x,\, \langle p\rangle_\Theta(\theta)] + \frac{\partial\theta}{\partial p}\,[x,\, \langle p\rangle_\Theta(\theta)]\,\eta\,(x,\theta)\end{aligned} \tag{8.235}$$

We have in pressure coordinates

$$\theta(x,p) = \langle\theta\rangle_P(p) + \theta'(x,p) \tag{8.236}$$

where θ' is small as long as η is small. Inserting this into the second term in (8.235), and neglecting all products of small terms, yields

$$\theta \approx \theta\,[x,\, \langle p\rangle_\Theta(\theta)] + \frac{\partial\langle\theta\rangle_P}{\partial p}\,(\langle p\rangle_\Theta)\,\eta\,(x,\theta) \tag{8.237}$$

Due to $\langle\eta\rangle_\Theta = 0$ averaging of this equation yields

$$\theta \approx \langle\theta\rangle_P\,[\langle p\rangle_\Theta(\theta)] \tag{8.238}$$

Inserting this into (8.235) we finally obtain

$$\eta(x,\theta) \approx \frac{\langle\theta\rangle_P\,[\langle p\rangle_\Theta(\theta)] - \theta\,[x,\, \langle p\rangle_\Theta(\theta)]}{\dfrac{\partial\langle\theta\rangle_P}{\partial p}\,[\langle p\rangle_\Theta(\theta)]} = -\frac{\theta'\,[x,\, \langle p\rangle_\Theta(\theta)]}{\dfrac{\partial\langle\theta\rangle_P}{\partial p}\,[\langle p\rangle_\Theta(\theta)]} \tag{8.239}$$

which, inserted into (8.234), leads to

$$\langle X\rangle_\Theta(\theta) \approx \left[\langle X\rangle_P - \frac{\langle\theta'\partial X/\partial p\rangle_P}{\partial\langle\theta\rangle_P/\partial p}\right][\langle p\rangle_\Theta(\theta)] = \left[\langle X\rangle_P - \frac{\langle\theta'\partial X'/\partial p\rangle_P}{\partial\langle\theta\rangle_P/\partial p}\right][\langle p\rangle_\Theta(\theta)] \tag{8.240}$$

and hence also

$$\langle v\rangle_\Theta(\theta) \approx \left[\langle v\rangle_P - \frac{\left\langle\theta'\partial v'/\partial p\right\rangle_P}{\partial\langle\theta\rangle_P/\partial p}\right][\langle p\rangle_\Theta(\theta)] \tag{8.241}$$

Now turn to the Stokes drift entering (8.232). From (8.239) and (8.233) follows

$$\frac{\partial p}{\partial\theta}(x,\theta) = \frac{\partial\langle p\rangle_\Theta}{\partial\theta}(\theta) - \frac{\partial}{\partial p}\left(\frac{\theta'}{\partial\langle\theta\rangle_P/\partial p}\right)\Bigg|_{[x,\langle p\rangle_\Theta(\theta)]}\frac{\partial\langle p\rangle_\Theta}{\partial\theta}(\theta) \tag{8.242}$$

and hence, with (8.219), the thickness

$$\sigma = -\frac{1}{g}\frac{\partial\langle p\rangle_\Theta}{\partial\theta}(\theta) + \frac{1}{g}\frac{\partial}{\partial p}\left(\frac{\theta'}{\partial\langle\theta\rangle_P/\partial p}\right)\Bigg|_{[x,\langle p\rangle_\Theta(\theta)]}\frac{\partial\langle p\rangle_\Theta}{\partial\theta}(\theta) \tag{8.243}$$

Obviously this can be decomposed into its mean

$$\langle\sigma\rangle_\Theta(\theta) = -\frac{1}{g}\frac{\partial\langle p\rangle_\Theta}{\partial\theta}(\theta) \tag{8.244}$$

and the fluctuation part

$$\sigma'(x,\theta) = -\langle\sigma\rangle_\Theta(\theta)\,\frac{\partial}{\partial p}\left(\frac{\theta'}{\partial\langle\theta\rangle_P/\partial p}\right)\Bigg|_{[x,\langle p\rangle_\Theta(\theta)]} \tag{8.245}$$

Now we insert $X = v'\sigma'$ into (8.234) neglect all cubic products of fluctuation terms, and obtain

$$\langle v'\sigma'\rangle_\Theta(\theta) \approx -\langle\sigma\rangle_\Theta(\theta)\left\langle v'\frac{\partial}{\partial p}\left(\frac{\theta'}{\partial\langle\theta\rangle_P/\partial p}\right)\right\rangle_P[\langle p\rangle_\Theta(\theta)] \tag{8.246}$$

We combine this with (8.241) in (8.232), which yields

$$\langle v\rangle_*(\theta) \approx \left[\langle v\rangle_P - \left\langle\frac{\partial v'}{\partial p}\frac{\theta'}{\partial\langle\theta\rangle_P/\partial p}\right\rangle_P - \left\langle v'\frac{\partial}{\partial p}\left(\frac{\theta'}{\partial\langle\theta\rangle_P/\partial p}\right)\right\rangle_P\right][\langle p\rangle_\Theta(\theta)] \tag{8.247}$$

Here we merge the two last terms. Moreover, we note that in a hydrostatic and stably stratified atmosphere there is an invertible mapping $p = \langle p\rangle_\Theta(\theta)$ so that the we obtain the estimate

$$\langle v\rangle_*(p) \approx \left[\langle v\rangle_P - \frac{\partial}{\partial p}\left\langle v'\frac{\theta'}{\partial\langle\theta\rangle_P/\partial p}\right\rangle_P\right](p) \tag{8.248}$$

of the mass-weighted mean meridional velocity in pressure coordinates.

For the final transformation to the geometric z-coordinate we make use of the synoptic scaling in quasigeostrophic theory. Then

$$p(x,z) = \overline{p}(z)\left[1 + \mathcal{O}(Ro^2)\right] \tag{8.249}$$

whence in general

$$\langle X\rangle_P\,[\overline{p}(z)] = \langle X\rangle_Z(z)\left[1 + \mathcal{O}(Ro^2)\right] \tag{8.250}$$

and therefore also

$$\frac{\partial \langle X \rangle_P}{\partial p}\left[\overline{p}(z)\right] = \frac{\partial z}{\partial \overline{p}}\frac{\partial \langle X \rangle_Z}{\partial z}(z)\left[1+\mathcal{O}(Ro^2)\right] = -\frac{1}{g\overline{\rho}}\frac{\partial \langle X \rangle_Z}{\partial z}(z)\left[1+\mathcal{O}(Ro^2)\right] \quad (8.251)$$

Moreover, we have in synoptic scaling

$$\theta(x,z) = \overline{\theta}(z)\left[1+\mathcal{O}(Ro^2)\right] \quad (8.252)$$

and hence

$$\frac{\partial \langle \theta \rangle_P}{\partial p}\left[\overline{p}(z)\right] = -\frac{1}{g\overline{\rho}}\frac{\partial \langle \theta \rangle_Z}{\partial z}(z)\left[1+\mathcal{O}(Ro^2)\right] = -\frac{1}{g\overline{\rho}}\frac{d\overline{\theta}}{dz}(z)\left[1+\mathcal{O}(Ro^2)\right] \quad (8.253)$$

so that (8.250) and (8.251) yield together with (8.248)

$$\langle v \rangle_* \approx \langle v \rangle_z - \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left\langle \overline{\rho}\frac{v'\theta'}{\partial\overline{\theta}/\partial z}\right\rangle\left[1+\mathcal{O}(Ro^2)\right] \quad (8.254)$$

Via (8.8) and (8.19) we finally obtain in good accuracy

$$\boxed{\langle v \rangle_* = \langle v \rangle_z - \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left\langle \frac{\overline{\rho}}{N^2}v'b'\right\rangle_z = \langle v \rangle^*} \quad (8.255)$$

under the condition that *the eddies are weak in amplitude and that fluctuations in the thermodynamic fields are small in comparison with the corresponding reference-atmosphere fields.* The latter holds, e.g., for synoptic-scale fluctuations. The mean mass streamfunction and residual streamfunction for the northern-hemisphere winter are shown in Figs. 8.8 and 8.9. The similarity is quite pronounced. Especially to be noted, one now has one big cell in each hemisphere that transports mass throughout the upper troposphere and the stratosphere from the tropics to polar latitudes. This explains the climatology of the ozone distribution.

8.4.4 Summary

The *wave impact on the mean flow* and the question of *zonal-mean tracer transport* through the atmosphere are tightly interwoven:

- The Eulerian zonal mean of the equations of motion shows that the *zonal-mean atmosphere is influenced by the convergence of the wave fluxes of momentum and buoyancy.* This perspective, however, leads to complex cancelations, e.g., between eddy buoyancy fluxes and heating. As opposed to that, the *residual circulation in the transformed Eulerian mean (TEM)* behaves so that *heating (cooling) alone will lead to upwelling (downwelling) of air masses.* Moreover, *climatological tracer distributions, e.g., of ozone, cannot be explained by the Eulerian mean circulation.*

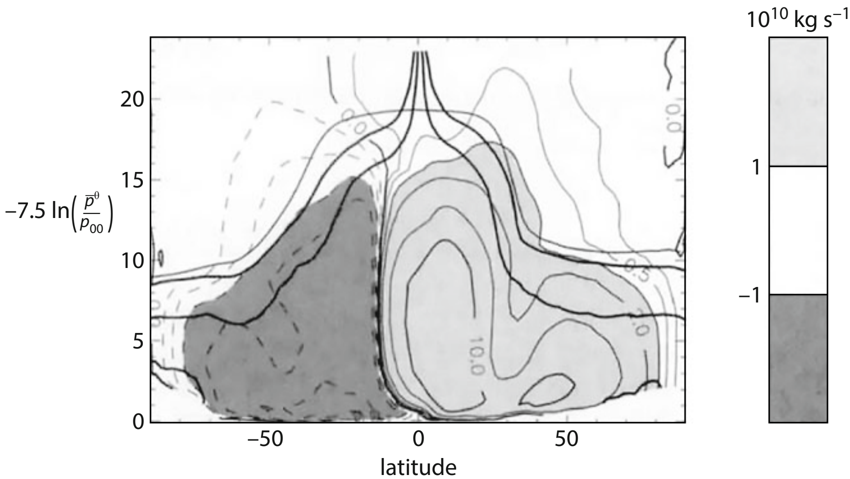

Fig. 8.8 Mass streamfunction in northern-hemisphere winter Juckes (2001).

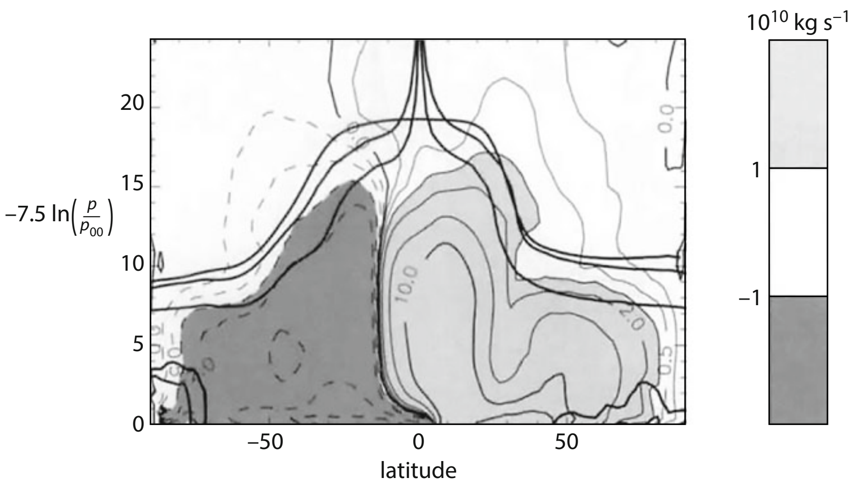

Fig. 8.9 Streamfunction of the residual-mean circulation in northern-hemisphere winter Juckes (2001).

- The reason for this discrepancy is the *Stokes drift* by which waves contribute to zonal-mean transport. It can be calculated most easily in *isentropic coordinates,* where *the zonal-mean transport is weighted by the transported mass.*
- One can show by corresponding transformations that the *mass-weighted circulation is nearly identical with the residual circulation.*

8.5 The Non-acceleration Theorem

The discussion up to this point suggests a general wave impact on the zonal mean of the atmosphere. It is an import caveat, however, that this influence is only given under certain circumstances. To show this, one begins with the conservation Eq. (8.14) for the quasi-geostrophic potential vorticity. Because the geostrophic wind has no divergence, we can write it in the flux form

$$\frac{\partial \pi}{\partial t} + \nabla \cdot (\mathbf{u}\pi) = D \tag{8.256}$$

The zonal average yields

$$\frac{\partial \langle \pi \rangle}{\partial t} + \frac{\partial}{\partial y}\langle v\pi \rangle = \langle D \rangle \tag{8.257}$$

In synoptic scaling within quasigeostrophic theory, however, $v = v' + \mathcal{O}(Ro)$ so that we find in good approximation.

$$\frac{\partial \langle \pi \rangle}{\partial t} + \frac{\partial}{\partial y}\langle v'\pi' \rangle = \langle D \rangle \tag{8.258}$$

Via the identity (8.112) between the Eliassen–Palm flux divergence and the meridional flux of potential vorticity, and the Eliassen–Palm relation (8.115), one obtains

$$\langle v'\pi' \rangle = \frac{1}{\overline{\rho}}\left(\frac{\overline{\rho}\langle D'\pi' \rangle}{\partial \langle \pi \rangle/\partial y} - \frac{\partial \mathcal{A}}{\partial t}\right) \tag{8.259}$$

which leads to

$$\frac{\partial \langle \pi \rangle}{\partial t} = \frac{\partial}{\partial y}\left(\frac{1}{\overline{\rho}}\frac{\partial \mathcal{A}}{\partial t} - \frac{\langle D'\pi' \rangle}{\partial \langle \pi \rangle/\partial y}\right) + \langle D \rangle \tag{8.260}$$

Additionally we know that

$$\pi = \nabla_h^2 \psi + f_0 + \beta y + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) \tag{8.261}$$

Together with (8.5) this gives

$$\frac{\partial}{\partial y}\frac{\partial \langle \pi \rangle}{\partial t} = -\frac{\partial^2}{\partial y^2}\frac{\partial \langle u \rangle}{\partial t} - \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\frac{\partial \langle u \rangle}{\partial t}\right) \tag{8.262}$$

Inserting this into the meridional derivative of (8.260) finally yields

$$\boxed{\left[\frac{\partial^2}{\partial y^2} + \frac{1}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}\frac{f_0^2}{N^2}\frac{\partial}{\partial z}\right)\right]\frac{\partial \langle u \rangle}{\partial t} = -\frac{\partial^2}{\partial y^2}\left(\frac{1}{\overline{\rho}}\frac{\partial \mathcal{A}}{\partial t} - \frac{\langle D'\pi' \rangle}{\partial \langle \pi \rangle/\partial y}\right) - \frac{\partial \langle D \rangle}{\partial y}} \tag{8.263}$$

This is an elliptic equation for the time derivate of the zonal-mean zonal wind. The latter vanishes if on the one hand the right-hand side does not contribute and on the other hand the zonal-mean zonal wind does not have a time derivative at the boundaries of the considered area (i.e., the β-channel). For the right-hand side to vanish we need $D = 0$ and $\partial\mathcal{A}/\partial t = 0$. This means there are no non-conservative forces and that the amplitudes of the waves are steady. Furthermore we recall that we have assumed at various instances of the derivation that the amplitudes of the waves are small so that the dynamics can be assumed to be linear. In summary the following theorem holds:

The non-acceleration theorem of Charney and Drazin: The zonal-mean zonal wind is steady in a domain if the following requirements are fulfilled:

- The wave amplitudes are small enough for linear dynamics to hold.
- The wave amplitudes are steady.
- There are no non-conservative forces.
- The zonal-mean zonal wind is steady on the domain boundaries. This last condition, however, does not apply on the sphere.

Under the conditions of this theorem we also know, moreover, from (8.259) that

$$\langle v'\pi' \rangle = 0 \tag{8.264}$$

Using (8.214) this results in

$$\langle v \rangle^* = 0 \tag{8.265}$$

because there are no non-conservative forces so that $\langle F \rangle = 0$. Then, however, the continuity Eq. (8.212) also implies

$$\langle w \rangle^* = 0 \tag{8.266}$$

Under the conditions of the non-acceleration theorem there is no residual circulation. Hence

$$\langle v \rangle - \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \left\langle \frac{\overline{\rho}}{N^2} v'b' \right\rangle = 0 \tag{8.267}$$

$$\langle w \rangle + \frac{1}{N^2} \frac{\partial}{\partial y} \langle v'b' \rangle = 0 \tag{8.268}$$

so that the waves exactly balance the Eulerian mean in the residual circulation. Moreover, (8.215) tells us that under the conditions of the non-acceleration theorem

$$\frac{\partial \langle b \rangle}{\partial t} = 0 \tag{8.269}$$

because the absence of non-conservative forces also implies $\langle Q \rangle = 0$. *Under the conditions of the non-acceleration theorem the zonal-mean potential temperature is steady as well. Therefore there generally is no interaction between waves and mean flow.*

Vice versa, a violation of the conditions of the theorem is required for the waves to influence the mean flow! This requires, e.g., time variance of the wave amplitudes or the influence of non-conservative processes. If, for instance,

- the wave amplitudes change in time,
- the waves break, i.e., they are dissipated by nonlinear effects, or
- friction or heating act,

then

- the mean flow can be accelerated or decelerated by the waves and
- a residual circulation exists.

These findings also help us understanding the residual circulation in the stratosphere: In the climatological mean the zonal-mean flow is steady so that the TEM Eqs. (8.214)–(8.215) become

$$\langle v\rangle^* = -\frac{1}{f_0}\left(\langle v'\pi'\rangle + \langle F\rangle\right) \tag{8.270}$$

$$\langle w\rangle^* = \frac{1}{N^2}\langle Q\rangle \tag{8.271}$$

As was to be expected, upwelling (downwelling) results from heating (cooling). For a better understanding of the meridional circulation we use (8.112) and hence obtain

$$\langle v\rangle^* = -\frac{1}{f_0}\left(\frac{1}{\overline{\rho}}\nabla\cdot\boldsymbol{\mathcal{F}} + \langle F\rangle\right) \tag{8.272}$$

Below the stratopause[4] the Eliassen–Palm flux divergence dominates on the right-hand side, due to Rossby waves propagating upward from the troposphere. Their group velocity is directed upwards as well so that, due to (8.154), the vertical component of the Eliassen–Palm flux $\mathcal{F}_z > 0$ is also positive. Whenever they are absorbed in the stratosphere, e.g., by wave breaking near critical layers, one gets $\partial\mathcal{F}_z/\partial z < 0$. Likewise the absorption of waves also propagating horizontally leads in general to a convergence of the Eliassen–Palm flux $\nabla\cdot\boldsymbol{\mathcal{F}} < 0$. Hence $f_0\langle v\rangle^* > 0$ so that the residual meridional velocity in the northern (southern) hemisphere is positive (negative); i.e., it is directed from the equator to the poles.

[4] In the mesosphere higher up small-scale gravity waves play the dominant role. In the present formulation their influence can be captured via corresponding forcing $\langle F\rangle \neq 0$ and heating $\langle Q\rangle \neq 0$.

8.6 Recommendations for Further Reading

Good textbook coverage of wave–mean flow interactions can be found in the books of Holton and Hakim (2013), Pedlosky (1987), and Vallis (2006), but especially worthwhile are the texts from Andrews et al. (1987) and Bühler (2009). The latter gives an excellent account of the generalized Lagrangian mean theory that has been introduced by Andrews and McIntyre (1978a, b), and that forms an important part of the basis of our understanding of the interactions between waves and mean flows, including the TEM and the non-acceleration theorem. Fundamental publications on meridional and vertical Rossby-wave propagation are Charney and Drazin (1961), Hoskins and Karoly (1981), and Karoly and Hoskins (1982). The Eliassen–Palm flux has been introduced by Eliassen and Palm (1961).