The Meridional Circulation

9

The knowledge gathered so far, on the dynamics of the atmosphere in general and on wave-mean-flow interaction in particular, will now be applied in a discussion of the mechanism of the general circulation of the atmosphere. The goal is to achieve a better understanding, beyond diagnostics, why circulation, zonal mean of wind and temperature, and waves arrange themselves as observed. Important open questions still remain, but the theory of atmospheric dynamics can explain a lot. We will here first sketch a few essentials of the empirical findings, then discuss the circulation in the tropics and finally we will consider the midlatitudes.

9.1 Some Essentials of the Empirical Basis

We first consider the radiation budget. Figure 9.1 shows the latitude-dependent power density of the zonal mean of the incoming solar radiation at the top of the atmosphere and the zonal mean of the outgoing infrared radiation. The two profiles do not match exactly. According to the Stefan–Boltzmann law, the outgoing atmospheric radiation L corresponds to a radiative temperature T so that $L = \sigma T^4$. Hence, the atmospheric radiative temperature is in polar regions warmer and in the tropics colder than expected from simple radiative equilibrium. Obviously there must be transport processes that transport thermal energy from the tropics to the polar regions. In addition to the oceans, the atmosphere has a considerable share in this effect. Here the direct atmospheric latitude–altitude–circulation contributes as well as synoptic-scale waves due to baroclinic instability.

The resulting latitude–altitude distribution of the zonally averaged potential temperature of the troposphere, together with the zonal-mean zonal wind, is shown in Fig. 9.2. Besides the general decrease in temperature from the equator to the poles one notes a strongly baroclinic zone in the subtropics, together with the corresponding thermal westerlies. The jet stream, however, extends further into the midlatitudes, where it has a more barotropic structure. Correspondingly the mid latitude surface winds are westerlies as well, while in

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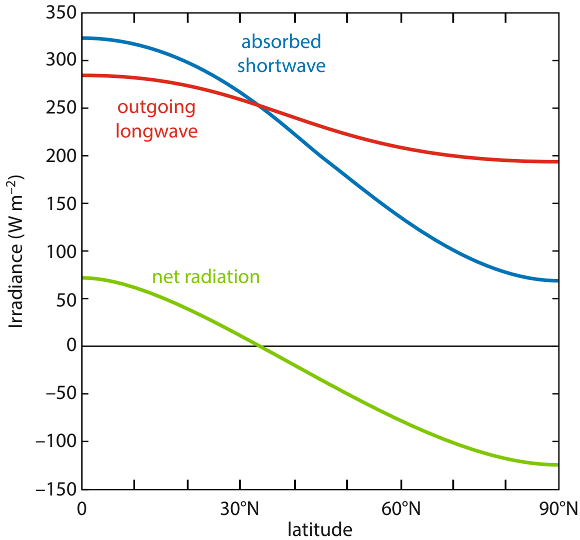

Fig. 9.1 Schematic representation of the latitudinal dependency of the zonal mean of the incoming solar radiation at the top of the atmosphere, the emitted infrared radiation, and the net radiation (incoming–outgoing) resulting from the two

the tropics and in the polar latitudes easterlies prevail. We also recall Fig. 8.7, showing the Eulerian-mean meridional circulation for northern-hemisphere winter. One can see the two Hadley cells in the tropics, with a much stronger cell on the winter side. This circulation is direct, i.e., in concordance with the thermal structure so that warm air masses rise and cold air masses sink. The Hadley circulation is flanked by the Ferrel cells that are thermally indirect. Obviously wave driving must be of significance here. In polar latitudes one can identify again weak direct circulation cells.

9.2 The Hadley Circulation

With regard to the circulation in the tropics we begin by discussing the zonally symmetric dynamics without waves and with symmetry between the northern and southern hemispheres. This is based on the work of Schneider (1977) as well as Held and Hou (1980). Next we will address the summer–winter–asymmetry and finally also consider the influence of waves.

9.2.1 The Basic Equations of a Model Without Wave Driving

We consider the primitive equations on a sphere, where we assume that there are no waves so that everything is zonally symmetric. Additionally we only look for steady solutions. As important dynamical aspects, we allow for turbulent friction and diffusion in the planetary boundary layer, and differential heating between the equator and the poles is included as well. Therefore we have

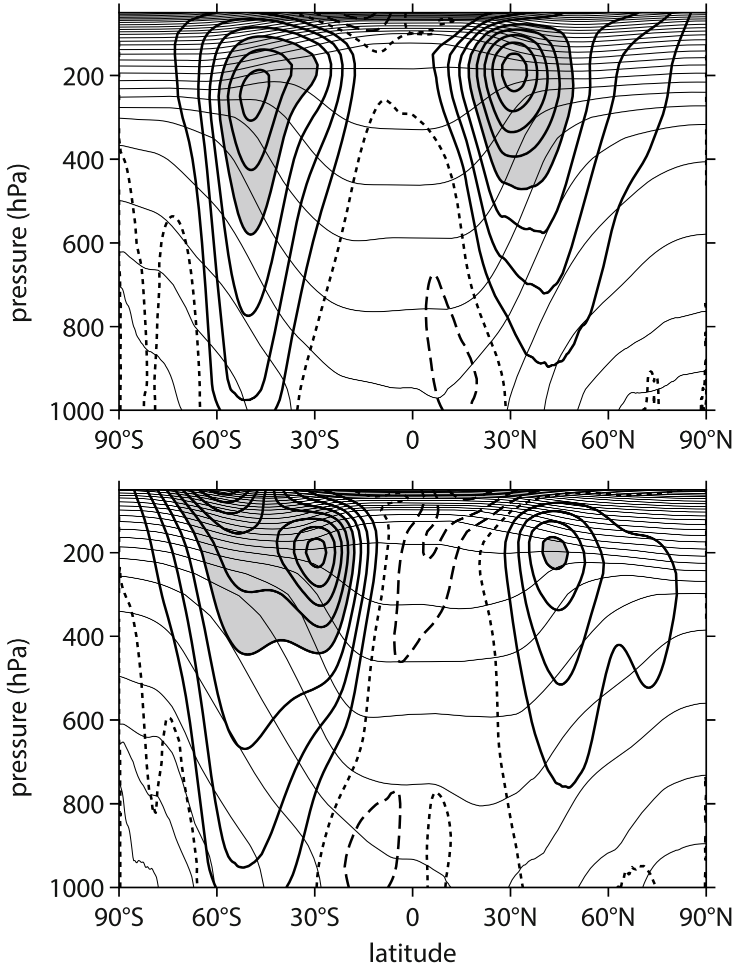

Fig. 9.2 The latitude–altitude distribution of the zonal mean of the zonal wind (contour interval 5 m/s, areas with >20 m/s are shaded) and the potential temperature (contour interval 10 K) in the troposphere in northern-hemisphere winter (top picture) and summer (bottom) in the ERA5 data (Hersbach et al., 2020)

$$\mathbf{v}\cdot\nabla u - fv - \frac{uv}{a}\tan\phi = \frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right) \tag{9.1}$$

$$\mathbf{v}\cdot\nabla v + fu + \frac{u^2}{a}\tan\phi = -\frac{1}{a\rho}\frac{\partial p}{\partial\phi} + \frac{\partial}{\partial z}\left(K\frac{\partial v}{\partial z}\right) \tag{9.2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{9.3}$$

$$\nabla\cdot(\rho\mathbf{v}) = 0 \tag{9.4}$$

$$\mathbf{v}\cdot\nabla\theta = \frac{\partial}{\partial z}\left(K\frac{\partial\theta}{\partial z}\right) - \frac{\theta-\theta_E}{\tau} \tag{9.5}$$

Due to the zonal symmetry we have

$$\mathbf{v}\cdot\nabla = \frac{v}{a}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z} \tag{9.6}$$

and for every scalar field α

$$\nabla\cdot(\alpha\mathbf{v}) = \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\, v\alpha) + \frac{\partial}{\partial z}(w\alpha) \tag{9.7}$$

Furthermore we have approximated the vertical turbulent fluxes by a simple flux-gradient relationship, with identical viscosity and diffusion coefficient K, and the heating by a relaxation ansatz

$$Q = -\frac{\theta-\theta_E}{\tau} \tag{9.8}$$

where $\theta_E(\phi, z)$ is the potential temperature of the radiative equilibrium and τ a relaxation time within which the atmosphere would adjust, in the absence of dynamics, to the radiative equilibrium. One can easily see that Q cools (heats) the atmosphere where the potential temperature is above (below) θ_E.

For reasons of simplicity we also use the Boussinesq approximation, i.e., we assume

$$\rho = \rho_0 + \tilde{\rho} \qquad |\tilde{\rho}| \ll \rho_0 \tag{9.9}$$

$$p = \overline{p} + \tilde{p} \qquad |\tilde{p}| \ll \overline{p} \tag{9.10}$$

$$\theta = \overline{\theta} + \tilde{\theta} \qquad |\tilde{\theta}| \ll \theta_0 \tag{9.11}$$

where $\overline{\theta}(z) = \theta_0 + \delta\overline{\theta}(z)$ with $|\delta\overline{\theta}| \ll \theta_0$. ρ_0 and θ_0 are constants, and ρ_0 and $\overline{p}(z)$ are in hydrostatic balance so that

$$\frac{d\overline{p}}{dz} = -\rho_0 g \tag{9.12}$$

As already discussed with regard to boundary layer theory, (9.2) and (9.3) then become

$$\mathbf{v} \cdot \nabla v + fu + \frac{u^2}{a} \tan\phi = -\frac{1}{a}\frac{\partial P}{\partial \phi} + \frac{\partial}{\partial z}\left(K \frac{\partial v}{\partial z}\right) \tag{9.13}$$

$$0 = -\frac{\partial P}{\partial z} + g\frac{\theta - \theta_0}{\theta_0} \tag{9.14}$$

with

$$P = \frac{\tilde{p}}{\rho_0} \tag{9.15}$$

The continuity Eq. (9.4) simplifies to

$$\nabla \cdot \mathbf{v} = 0 \tag{9.16}$$

Therefore all advection terms can be rewritten as flux terms, e.g.,

$$\mathbf{v} \cdot \nabla u = \nabla \cdot (u\mathbf{v}) \tag{9.17}$$

In summary one obtains

$$\nabla \cdot (\mathbf{v}u) - fv - \frac{uv}{a} \tan\phi = \frac{\partial}{\partial z}\left(K \frac{\partial u}{\partial z}\right) \tag{9.18}$$

$$\nabla \cdot (\mathbf{v}v) + fu + \frac{u^2}{a} \tan\phi = -\frac{1}{a}\frac{\partial P}{\partial \phi} + \frac{\partial}{\partial z}\left(K \frac{\partial v}{\partial z}\right) \tag{9.19}$$

$$0 = -\frac{\partial P}{\partial z} + g\frac{\theta - \theta_0}{\theta_0} \tag{9.20}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{9.21}$$

$$\nabla \cdot (\mathbf{v}\theta) = \frac{\partial}{\partial z}\left(K \frac{\partial \theta}{\partial z}\right) - \frac{\theta - \theta_E}{\tau} \tag{9.22}$$

In order to solve the equations we need *boundary conditions*. At the *upper boundary* ($z = H$) we assume that there is no vertical motion, i.e.,

$$z = H: \qquad w = 0 \tag{9.23}$$

and therefore all vertical turbulent fluxes, for example $\langle u'w' \rangle$, vanish as well. Within the flux-gradient approximation these are proportional to the vertical gradient of the mean fields, e.g.,

$$\langle u'w' \rangle = -K\frac{\partial u}{\partial z} \tag{9.24}$$

Hence one has

$$z = H: \qquad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial \theta}{\partial z} = 0 \tag{9.25}$$

At the *lower boundary* ($z = 0$) we similarly assume

$$z = 0: \qquad w = 0 \tag{9.26}$$

and we also neglect the turbulent heat flux through the ground, i.e.,

$$z = 0: \qquad \frac{\partial \theta}{\partial z} = 0 \tag{9.27}$$

However, ignoring the turbulent momentum flux is not possible. The turbulence of the boundary layer communicates the effect of the molecular viscosity so that momentum is transferred from the solid earth to the atmosphere. In doing so it tends to decelerate the laminar flow. Hence we can simply assume

$$\langle u'w' \rangle = -K \frac{\partial u}{\partial z} = -Cu \tag{9.28}$$

with $C > 0$ a constant friction coefficient. Hence one uses

$$z = 0: \qquad K \frac{\partial u}{\partial z} = Cu \qquad K \frac{\partial v}{\partial z} = Cv \tag{9.29}$$

9.2.2 A Solution Without Meridional Circulation

Without turbulent fluxes, i.e., when

$$K = 0 \tag{9.30}$$

there is a simple solution without meridional circulation,

$$v = w = 0 \tag{9.31}$$

and without surface winds,

$$u(z = 0) = 0 \tag{9.32}$$

that is in perfect radiative balance, i.e.,

$$\theta = \theta_E \tag{9.33}$$

This solution trivially satisfies the zonal-momentum equation, the continuity equation and the entropy equation. The meridional and the vertical-momentum equations become

$$fu + \frac{u^2}{a} \tan \phi = -\frac{1}{a} \frac{\partial P}{\partial \phi} \tag{9.34}$$

$$0 = -\frac{\partial P}{\partial z} + g \frac{\theta_E - \theta_0}{\theta_0} \tag{9.35}$$

i.e., it is hydrostatic and in a generalized geostrophic balance. The vertical derivative of (9.34) yields, with the aid of (9.35),

$$\frac{\partial}{\partial z}\left(fu + \frac{u^2}{a}\tan\phi\right) = -\frac{1}{a}\frac{\partial}{\partial\phi}\left(g\frac{\theta_E - \theta_0}{\theta_0}\right) = -\frac{g}{a\theta_0}\frac{\partial\theta_E}{\partial\phi} \tag{9.36}$$

which is a generalization of the thermal-wind relation. Because there are no surface winds, vertical integration of (9.36) yields

$$fu + \frac{u^2}{a}\tan\phi = -\frac{g}{a\theta_0}\int_0^z dz'\frac{\partial\theta_E}{\partial\phi} \tag{9.37}$$

This quadratic equation in u has only one solution in agreement with (9.32):

$$u = \Omega a\cos\phi\left[\left(1 - \frac{g}{\Omega^2 a^2\sin\phi\cos\phi}\int_0^z dz'\frac{\partial\theta_E}{\partial\phi}\right)^{1/2} - 1\right] \tag{9.38}$$

The potential temperature of the radiative equilibrium decreases from the equator to the poles, and therefore

$$\frac{1}{\sin\phi}\frac{\partial\theta_E}{\partial\phi} < 0 \tag{9.39}$$

For this we assume for the latitude dependence of θ_E that its derivative vanishes at the equator. Otherwise there would be a singularity at the equator so that the neglect of turbulent friction would not be permissible. Nonetheless, this would still admit equality in (9.39). Equilibrium temperatures of interest for us are, however, of the form (9.83) so that strict inequality holds. Hence we have everywhere above the ground

$$\frac{g}{\Omega^2 a^2\sin\phi\cos\phi}\int_0^z dz'\frac{\partial\theta_E}{\partial\phi} < 0 \tag{9.40}$$

and therefore

$$u > 0 \tag{9.41}$$

This is also the case at the equator, i.e., the atmosphere is rotating faster than the earth. It is in a state of *superrotation*. Moreover, the zonal wind is everywhere increasing with altitude, i.e.,

$$\frac{\partial u}{\partial z} = -\frac{g}{2\Omega a\sin\phi}\frac{\partial\theta_E}{\partial\phi}\left(1 - \frac{g}{\Omega^2 a^2\sin\phi\cos\phi}\int_0^z dz'\frac{\partial\theta_E}{\partial\phi}\right)^{-1/2} > 0 \tag{9.42}$$

9.2.3 Hide's Theorem

One could expect that it is possible to obtain a solution for small K by using the one above as a starting point and expanding around it. Surprisingly, however, solutions for $K \neq 0$ do not converge to superrotation without meridional circulation as $K \to 0$. This results from the conservation of angular-momentum. The corresponding equation can be obtained by first rewriting the zonal-momentum Eq. (9.18), using the fact that the wind field has no divergence,

$$\mathbf{v} \cdot \nabla u - 2\Omega \sin\phi v - \frac{uv}{a} \tan\phi = \frac{\partial}{\partial z}\left(K \frac{\partial u}{\partial z}\right) \tag{9.43}$$

Multiplying this with $a \cos\phi$ and using (9.6) yields

$$\frac{v}{a}\frac{\partial m}{\partial \phi} + w\frac{\partial m}{\partial z} = \frac{\partial}{\partial z}\left(K \frac{\partial m}{\partial z}\right) \tag{9.44}$$

with

$$m = \Omega a^2 \cos^2\phi + ua\cos\phi \tag{9.45}$$

the mass-specific density of the axial angular-momentum component. Using (9.6) and again the non-divergence of the wind field leads the following conservation equation

$$0 = -\nabla \cdot (m\mathbf{v}) + \frac{\partial}{\partial z}\left(K \frac{\partial m}{\partial z}\right) \tag{9.46}$$

Using this one can show that the maximum of m must be located at the lower boundary of the atmosphere, and that it must be in a region with easterlies $u < 0$. Hence the global maximum of m must satisfy (9.45)

$$m_0 < \Omega a^2 \tag{9.47}$$

so that

$$u = \frac{m - \Omega a^2 \cos^2\phi}{a\cos\phi} < \frac{m_0 - \Omega a^2 \cos^2\phi}{a\cos\phi} < \Omega a \frac{\sin^2\phi}{\cos\phi} \tag{9.48}$$

This requires *easterlies at the equator,* in contrast to superrotation. Moreover, at the ground close to the maximum of m one must have $\partial u/\partial z < 0$ which cannot approach (9.42) in a continuous manner as $K \to 0$.

We first show that the maximum cannot be located in the interior of the atmosphere, by demonstrating that this would lead to a contradiction. If there were a maximum m_0 in the inner zone of the atmosphere, there would also be, in sufficient vicinity, a closed contour C, around the location of the maximum, where the angular-momentum density has a constant value $m = m_C < m_0$. This is shown in Fig. 9.3. Integrating (9.46) over the enclosed area S_C yields

$$0 = -\int_{S_C} dS \nabla \cdot (m\mathbf{v}) + \int_{S_C} dS \frac{\partial}{\partial z}\left(K \frac{\partial m}{\partial z}\right) \tag{9.49}$$

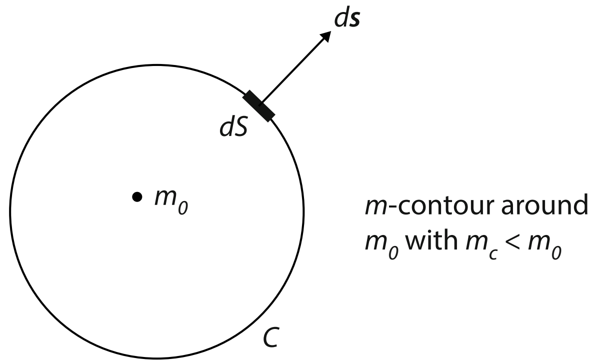

Fig. 9.3 A closed contour C with a constant angular-momentum density m_C, that includes a maximum m_0. The vector $d\mathbf{s}$ is parallel to the outward directed normal vector, and its magnitude agrees with the length of the curve element ds

Applying Gauss' theorem twice, and using the non-divergence of the flow field, yields for the first integral

$$\int_{S_C} dS \nabla \cdot (m\mathbf{v}) = \oint_C d\mathbf{s} \cdot m\mathbf{v} = m_C \oint_C d\mathbf{s} \cdot \mathbf{v} = m_C \int_{S_C} dS \nabla \cdot \mathbf{v} = 0 \tag{9.50}$$

With the help of Fig. 9.4 one can see that the second integral is

$$\int_{S_C} dS \frac{\partial}{\partial z}\left(K\frac{\partial m}{\partial z}\right) = \int_{S_C} dz d\phi \, a \frac{\partial}{\partial z}\left(K\frac{\partial m}{\partial z}\right) = \int_{\phi_1}^{\phi_2} d\phi \, a \left[K\frac{\partial m}{\partial z}\right]_{z_1(\phi)}^{z_2(\phi)} < 0 \tag{9.51}$$

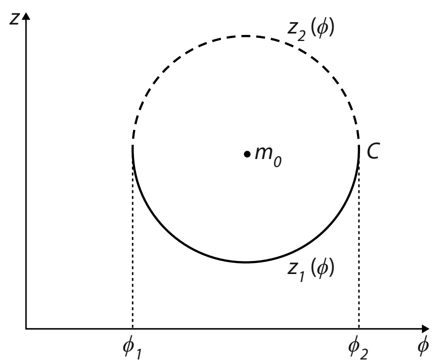

Fig. 9.4 A helpful segmentation of the contour C in Fig. 9.3, into parts that can be written as $z_i(\phi)$

because one has

$$\left.\frac{\partial m}{\partial z}\right|_{z2} \leq 0 \tag{9.52}$$

$$\left.\frac{\partial m}{\partial z}\right|_{z1} \geq 0 \tag{9.53}$$

with unequality dominating, if C is close enough to the maximum m_0. Obviously (9.49), (9.50), and (9.51) combined generate a contradiction, so that the maximum of m cannot be located in the interior of the atmosphere. Likewise one can show that the maximum cannot be located at the upper boundary. As shown in Fig. 9.5 there would be a contour C close enough to the maximum, that together with the upper boundary would enclose the maximum. Same way as above integration would then yield

$$0 = \int\limits_{S_C} dS \frac{\partial}{\partial z}\left(K\frac{\partial m}{\partial z}\right) = \int\limits_{\phi_1}^{\phi_2} d\phi\ a \left[K\frac{\partial m}{\partial z}\right]_{z_1(\phi)}^{H} \tag{9.54}$$

Because of the upper boundary condition (9.25) we have at $z = H$

$$K\frac{\partial m}{\partial z} = K a \cos\phi \frac{\partial u}{\partial z} = 0 \tag{9.55}$$

so that

$$0 = -\int\limits_{\phi_1}^{\phi_2} d\phi\ a\ K\left.\frac{\partial m}{\partial z}\right|_{z_1(\phi)} \tag{9.56}$$

Because for a contour close enough to m_0

$$\left.\frac{\partial m}{\partial z}\right|_{z_1(\phi)} \geq 0 \tag{9.57}$$

with prevailing inequality, this leads to a contradiction as well.

A maximum at the lower boundary, however, is possible. According to Fig. 9.6 the corresponding integral yields

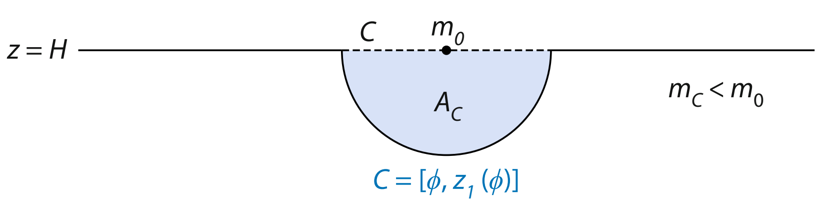

Fig. 9.5 A contour C with constant angular-momentum density m_C surrounding together with the upper boundary a maximum m_0 at the upper boundary of the atmosphere

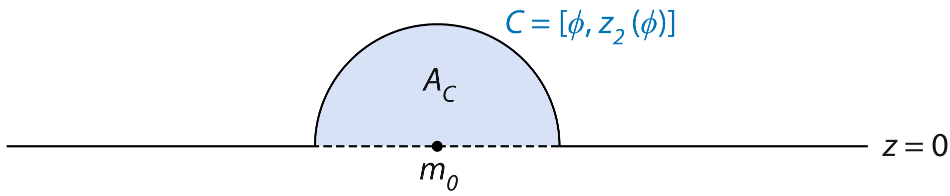

Fig. 9.6 A contour C with constant angular-momentum density m_C, that encloses together with the lower boundary a maximum at the lower boundary of the atmosphere

$$0 = \int\limits_{S_C} dS \frac{\partial}{\partial z}\left(K\frac{\partial m}{\partial z}\right) = \int\limits_{\phi_1}^{\phi_2} d\phi\ a \left[K\frac{\partial m}{\partial z}\right]_0^{z_2(\phi)} \tag{9.58}$$

Due to the lower boundary condition (9.29) one has

$$K\frac{\partial m}{\partial z}\bigg|_{z=0} = a\cos\phi K\frac{\partial u}{\partial z}\bigg|_{z=0} = a\cos\phi C u\bigg|_{z=0} \tag{9.59}$$

so that

$$0 = \int\limits_{\phi_1}^{\phi_2} d\phi\ a\left[K\frac{\partial m}{\partial z}\bigg|_{z_2} - a\cos\phi C u\bigg|_{z=0}\right] \tag{9.60}$$

This relation can be satisfied. However, because sufficiently close to the location of the maximum

$$\frac{\partial m}{\partial z}\bigg|_{z_2} \le 0 \tag{9.61}$$

with prevailing inequality, one then has close to this location $u < 0$. Hence in the limit $K \to 0$ no continuous transition to the superrotating solution would be possible.

9.2.4 A Simplified Description of the Hadley Cell

As shown by Held and Hou (1980), despite the complications discussed above there is a good analytical approach to the Hadley circulation that can be verified using numerical solutions. It is assumed that one cell extends from the equator up to the latitude ϕ_H, as sketched in Fig. 9.7. Five basic assumptions are used, to be supplemented by two additional ones later on:

i) In the upper branch near $z = H$ the turbulent viscosity does not affect the conservation of angular-momentum significantly, so that one has

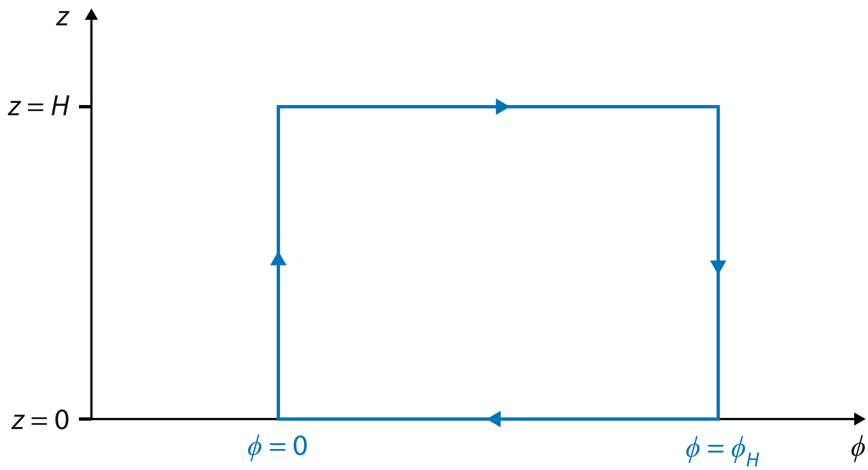

Fig. 9.7 Basic geometry of a Hadley cell adapted from Held and Hou (1980)

$$z = H: \qquad 0 = \nabla \cdot (m\mathbf{v}) \tag{9.62}$$

Because the flow field is non-divergent, and because of the upper boundary condition $w = 0$ at $z = H$, this can also be written

$$\frac{v}{a}\frac{\partial m}{\partial \phi} = 0 \tag{9.63}$$

or

$$\boxed{z = H: \qquad \frac{\partial m}{\partial \phi} = 0} \tag{9.64}$$

Hence the angular-momentum is constant along the upper branch of the Hadley cell, or

$$z = H: \qquad m = m|_{\phi=0} \tag{9.65}$$

ii) Advection and turbulent friction can be neglected in the meridional momentum equation, so that the same generalized geostrophic balance holds as in the consideration of the radiative equilibrium:

$$\boxed{fu + \frac{u^2}{a}\tan\phi = -\frac{1}{a}\frac{\partial P}{\partial \phi}} \tag{9.66}$$

iii) The zonal surface winds are significantly weaker than the winds at the upper boundary of the Hadley cell, i.e.,

$$\boxed{|u||_{z=0} \ll |u||_{z=H}} \tag{9.67}$$

iv) Turbulent diffusion does not play a decisive role in the entropy equation, so that one can write it

$$\boxed{\nabla \cdot (\mathbf{v}\theta) = -\frac{\theta - \theta_E}{\tau}} \tag{9.68}$$

v) Finally we assume that the circulation is *symmetric with respect to the equator* so that no mass crosses the equator:

$$\phi = 0: \qquad v = 0 \tag{9.69}$$

Because turbulence effects are neglected in most places, we essentially obtain a solution for $K \to 0$, while solutions for $K > 0$ are only possible by numerical methods.

Assumption i) yields the zonal wind in the upper branch of the Hadley cell, because by inserting (9.45) into (9.65) one obtains

$$z = H: \qquad \Omega a^2 \cos^2\phi + ua\cos\phi = \Omega a^2 + a\, u|_{\phi=0} \tag{9.70}$$

Because air masses originate at the equator from the rising branch of the Hadley cell, one can use assumption iii) to neglect $u|_{\phi=0}$, leading to

$$\boxed{z = H: \qquad u = u_M = \Omega a \frac{\sin^2\phi}{\cos\phi}} \tag{9.71}$$

Due to the conservation of angular-momentum, the zonal wind increases strongly from the equator to the midlatitudes. As will be explained below, one obtains outside the Hadley cell the above discussed radiative-equilibrium solution so that a jet stream results, as sketched in Fig. 9.8. Figure 9.9 shows the analytical jet stream solution together with numerical results for different turbulent viscosities. The corresponding latitude–altitude dependency is shown in Fig. 9.10.

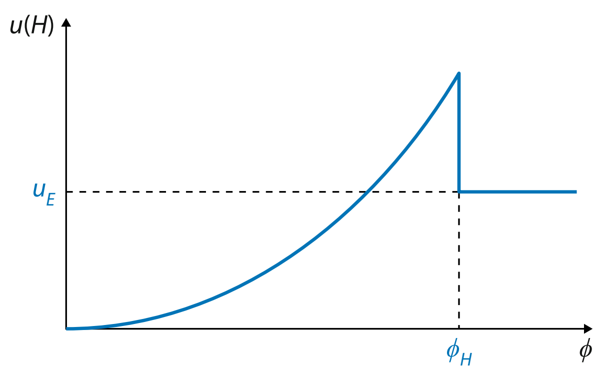

Fig. 9.8 Latitude dependence of the zonal wind in the upper troposphere according to the simplified model by Held and Hou (1980)

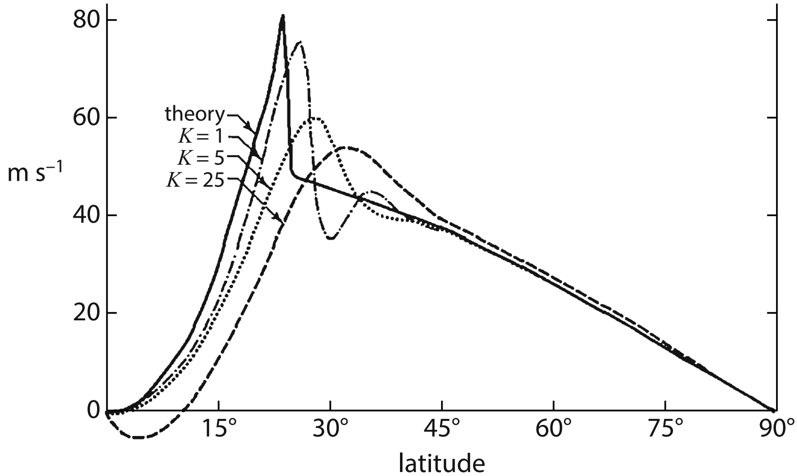

Fig. 9.9 Latitude dependence of the zonal wind in the upper troposphere, in the simplified model by Held and Hou (1980), together with numerical results for different turbulent viscosities (K), reprinted from Held and Hou (1980)

Due to assumption ii) one has the generalized geostrophic balance (9.66). The difference between the equilibria at the upper and lower boundaries, respectively, is

$$f\left[u(H)-u(0)\right]+\frac{\tan\phi}{a}\left[u^2(H)-u^2(0)\right]=-\frac{1}{a}\frac{\partial}{\partial\phi}\left[P(H)-P(0)\right] \tag{9.72}$$

The corresponding right-hand side is obtained by vertical integration of the hydrostatic balance

$$\frac{\partial P}{\partial z}=g\frac{\theta-\theta_0}{\theta_0} \tag{9.73}$$

leading to

$$\frac{1}{H}\left[P(H)-P(0)\right]=g\frac{\{\theta\}-\theta_0}{\theta_0} \tag{9.74}$$

where for an arbitrary field X

$$\{X\}=\frac{1}{H}\int_0^H dzX \tag{9.75}$$

denotes the vertical mean. Further applying the approximation (9.67) of weak surface winds and the latitude independence of θ_0 gives

$$z=H: \qquad fu+\frac{\tan\phi}{a}u^2=-\frac{gH}{a\theta_0}\frac{\partial\{\theta\}}{\partial\phi} \tag{9.76}$$

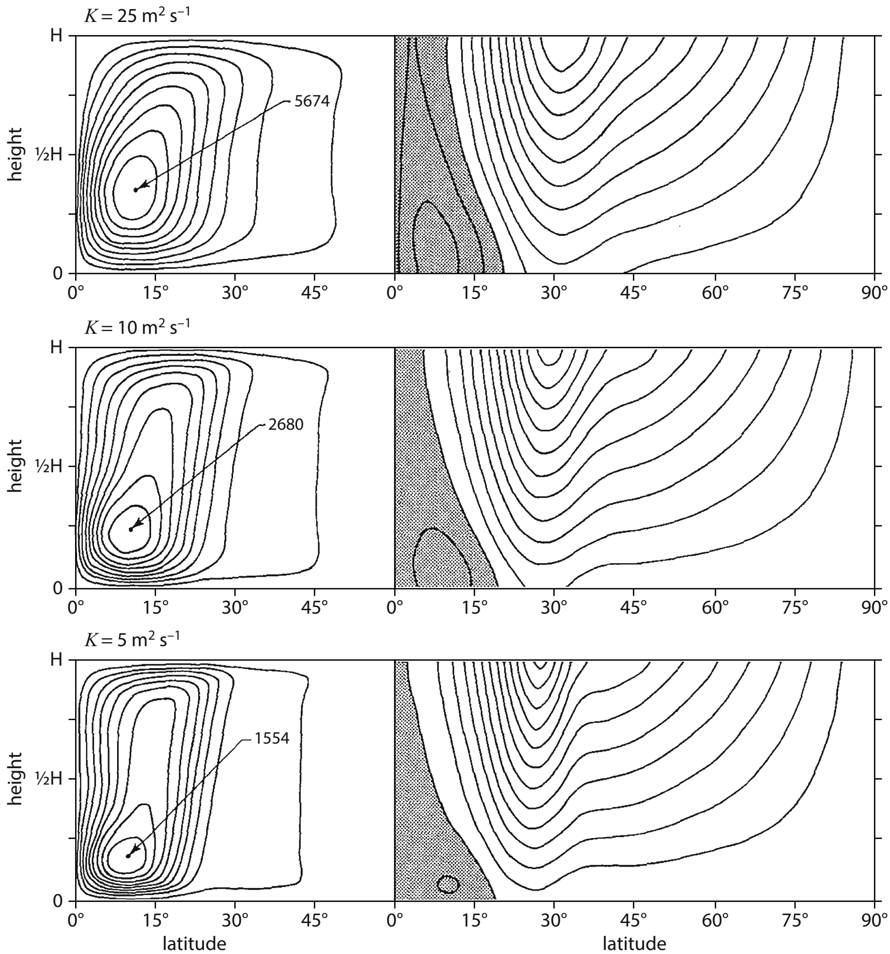

Fig. 9.10 Numerical results for the latitude–altitude dependence of the mass streamfunction (left) and the zonal wind (right) in the simplified model of Held and Hou (1980)

Inserting (9.71) and $f = 2\Omega \sin\phi$ yields

$$\frac{\Omega^2 a}{2} \frac{\partial}{\partial \phi} \frac{\sin^4 \phi}{\cos^2 \phi} = -\frac{gH}{a\theta_0} \frac{\partial \{\theta\}}{\partial \phi} \tag{9.77}$$

Integrating this relation over the latitude finally results in

$$\boxed{\frac{\{\theta\}(\phi) - \{\theta\}(0)}{\theta_0} = -\frac{\Omega^2 a^2}{2gH} \frac{\sin^4 \phi}{\cos^2 \phi}} \tag{9.78}$$

Up to this point the latitude ϕ_H of the Hadley cell as well as the mean potential temperature $\{\theta\}(0)/\theta_0$ at the equator remain undetermined. Moreover, no proof has been given yet that outside the Hadley cell one has radiative balance so that the zonal wind is given by the corresponding thermal wind (9.38). The latter can be deduced from (9.68), because outside of the Hadley cells $v = w = 0$ so that

$$\theta = \theta_E \tag{9.79}$$

Combined with (9.76) this yields the desired result. The latitude ϕ_H can be found by assuming a meaningful profile for θ_E as well as continuity in the potential temperature so that

$$\phi = \phi_H : \qquad \{\theta\} = \{\theta_E\} \tag{9.80}$$

Because by definition the normal velocity component vanishes at the boundaries of the Hadley cell, see Fig. 9.11, the integration of the entropy Eq. (9.68) over the total cross section of the cell yields, using Gauss' theorem,

$$0 = -\frac{1}{\tau}\int\limits_{S_H} dS(\theta - \theta_E) \tag{9.81}$$

This leads to

$$\int\limits_0^{\phi_H} d\phi\; a\,\{\theta\} = \int\limits_0^{\phi_H} d\phi\; a\,\{\theta_E\} \tag{9.82}$$

The area under the curve of the vertically averaged potential temperature is the same as in radiative balance, as shown in Fig. 9.12. At this point an explicit spatial dependency of the radiative-equilibrium potential temperature is needed. A good compromise between realism and simplicity is

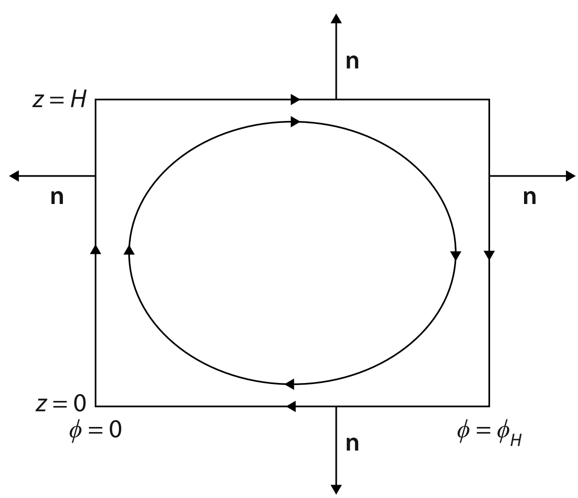

Fig. 9.11 Velocity field within a Hadley cell, with vanishing normal component at its boundaries

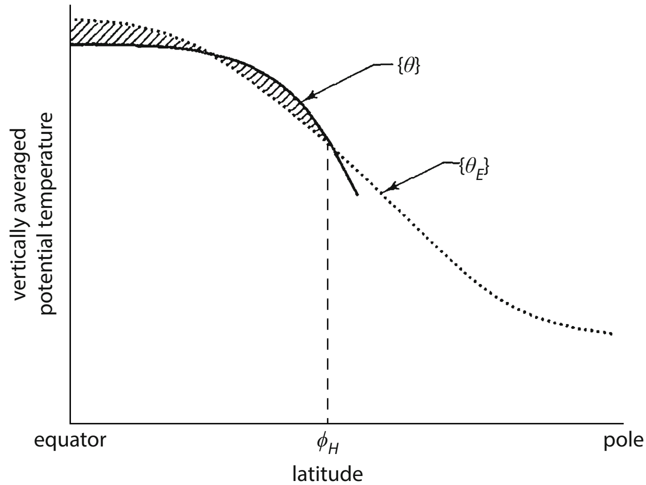

Fig. 9.12 Area under the curve of latitude dependence of the vertically averaged potential temperature is the same as in radiative balance. Therefore the shaded areas agree as well. Reprinted from (Held and Hou, 1980)

$$\frac{\theta_E}{\theta_0} = 1 - \frac{2}{3}\Delta_H P_2(\sin\phi) + \Delta_v\left(\frac{z}{H} - \frac{1}{2}\right) \tag{9.83}$$

Here Δ_H is the fraction by which the potential temperature differs between equator and pole, whereas Δ_v describes the fractional potential-temperature change between ground and top of the troposphere. Finally,

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \tag{9.84}$$

is a Legendre polynomial. Because one can assume in the tropics

$$|\phi| \ll 1 \tag{9.85}$$

(9.83) can be approximated well by

$$\frac{\theta_E}{\theta_0} = 1 + \frac{\Delta_H}{3} - \Delta_H\phi^2 + \Delta_v\left(\frac{z}{H} - \frac{1}{2}\right) \tag{9.86}$$

whence

$$\frac{\{\theta_E\}}{\theta_0} = 1 + \frac{\Delta_H}{3} - \Delta_H\phi^2 \tag{9.87}$$

Likewise applying the low-latitude approximation to (9.78) yields

$$\frac{\{\theta\}}{\theta_0} = \frac{\{\theta\}(0)}{\theta_0} - \frac{\Omega^2 a^2}{2gH}\phi^4 \tag{9.88}$$

Inserting both approximations into (9.80) and (9.82) leads to

$$\frac{\{\theta\}(0)}{\theta_0} - \frac{\Omega^2 a^2}{2gH}\phi_H^4 = \frac{\{\theta_E\}(0)}{\theta_0} - \Delta_H \phi_H^2 \tag{9.89}$$

$$\frac{\{\theta\}(0)}{\theta_0} - \frac{\Omega^2 a^2}{10gH}\phi_H^4 = \frac{\{\theta_E\}(0)}{\theta_0} - \frac{\Delta_H}{3}\phi_H^2 \tag{9.90}$$

where

$$\frac{\{\theta_E\}(0)}{\theta_0} = 1 + \frac{\Delta_H}{3} \tag{9.91}$$

The difference (9.90)–(9.89) gives

$$\frac{2}{5}\frac{\Omega^2 a^2}{gH}\phi_H^4 = \frac{2}{3}\Delta_H \phi_H^2 \tag{9.92}$$

or

$$\boxed{\phi_H = \sqrt{\frac{5}{3}R}} \tag{9.93}$$

with

$$\boxed{R = \frac{gH\Delta_H}{\Omega^2 a^2}} \tag{9.94}$$

The horizontal extent of the Hadley circulation increases with the baroclinicity of the atmosphere, while it decreases with the radius and the rotation frequency of the planet. For the earth a reasonable value for the baroclinicity is $\Delta_H = 1/6$, which yields $\phi_H \approx 20°$. Inserting (9.93) into (9.89) finally gives

$$\boxed{\frac{\{\theta\}(0)}{\theta_0} = \frac{\{\theta_E\}(0)}{\theta_0} - \frac{5}{18}R\Delta_H} \tag{9.95}$$

The difference between the two vertical-mean potential temperatures at the equator increases with the horizontal extent of the Hadley cell, because the strength of the circulation increases, and hence also the heat transport.

The latitude dependence of the vertical-mean heat flux can be obtained by reconsidering the entropy equation

$$\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\, v\theta) + \frac{\partial}{\partial z}(w\theta) = -\frac{\theta - \theta_E}{\tau} \tag{9.96}$$

Integrating this equation in the vertical and using the boundary conditions (9.23) and (9.26) leads to

$$\frac{1}{H}\int_0^H dz \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\, v\theta) = -\frac{\{\theta\}-\{\theta_E\}}{\tau} \tag{9.97}$$

Here (9.88), (9.87), and (9.95) yield in the low-latitude approximation

$$\frac{\{\theta\}-\{\theta_E\}}{\theta_0} = \frac{\{\theta\}(0)-\{\theta_E\}(0)}{\theta_0} - \frac{\Omega^2 a^2}{2gH}\phi^4 + \Delta_H\phi^2 = -\frac{5}{18}R\Delta_H - \frac{\Omega^2 a^2}{2gH}\phi^4 + \Delta_H\phi^2 \tag{9.98}$$

so that

$$\frac{1}{a}\frac{\partial}{\partial\phi}\left(\frac{1}{H}\int_0^H dz\, v\theta\right) = \frac{\theta_0}{\tau}\left(\frac{5}{18}R\Delta_H + \frac{\Omega^2 a^2}{2gH}\phi^4 - \Delta_H\phi^2\right) \tag{9.99}$$

where we have used the low-latitude approximation $\cos\phi \approx 1$. At the equator we have $v = 0$ so that

$$\left.\frac{1}{H}\int_0^H dz\, v\theta\right|_{\phi=0} = 0 \tag{9.100}$$

Hence, integrating in latitude from 0 to ϕ results in

$$\frac{1}{a}\frac{1}{H}\int_0^H dz\, v\theta = \frac{\theta_0}{\tau}\left(\frac{5}{18}R\Delta_H\phi + \frac{\Omega^2 a^2}{10gH}\phi^5 - \frac{\Delta_H}{3}\phi^3\right) \tag{9.101}$$

which can easily be reformulated as

$$\boxed{\frac{1}{\theta_0}\int_0^H dz\, v\theta = \frac{5}{18}\left(\frac{5}{3}\right)^{1/2}\frac{Ha\Delta_H}{\tau}R^{3/2}\left[\frac{\phi}{\phi_H} - 2\left(\frac{\phi}{\phi_H}\right)^3 + \left(\frac{\phi}{\phi_H}\right)^5\right]} \tag{9.102}$$

This result, together with numerical results for $K > 0$, is shown in Fig. 9.13.

To estimate the surface winds we need two additional basic assumptions. We first observe, however, that the vertical integral of the continuity equation

$$\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\, v) + \frac{\partial w}{\partial z} = 0 \tag{9.103}$$

with the boundary conditions (9.23) and (9.26) results in

$$\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\int dz\, v\right) = 0 \tag{9.104}$$

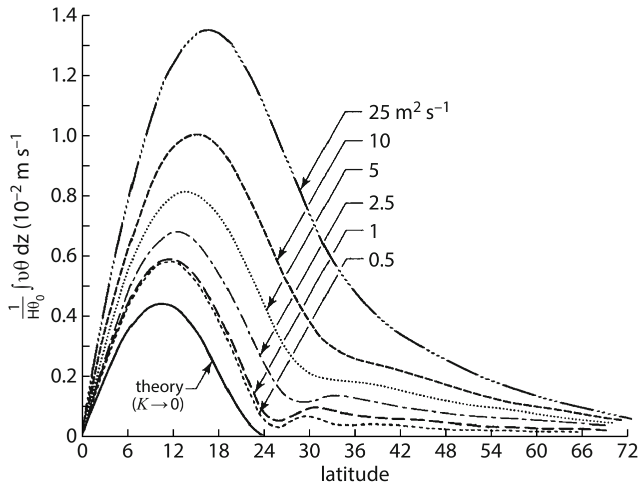

Fig. 9.13 Latitude dependence of the vertical-mean meridional heat transport. Shown are the analytical result for $K = 0$ and numerical results for $K > 0$, according to Held and Hou (1980)

But according to (9.69) there is no mass flux across the equator, i.e.,

$$\phi = 0: \qquad \int_0^H dz\, v = 0 \tag{9.105}$$

so that one has in general

$$\int_0^H dz\, v = 0 \tag{9.106}$$

Hence the poleward mass flux in the upper branch of a Hadley cell is balanced exactly by the equatorward flux in the lower branch. We now specify the two additional assumptions needed for the determination of the surface winds:

vi) As sketched in Fig. 9.14 the meridional flux is confined to thin layers near the top and the bottom boundaries. Following from the discussion above the two fluxes are of opposite sign and equal size:

$$V|_{z=H} = v(H)\Delta z(H) = -\ V|_{z=0} = -v(0)\Delta z(0) \tag{9.107}$$

vii) Furthermore we assume that the static stability is not altered significantly by the circulation, i.e.,

$$\frac{\theta|_{z=H} - \theta|_{z=0}}{\theta_0} = \Delta_v \tag{9.108}$$

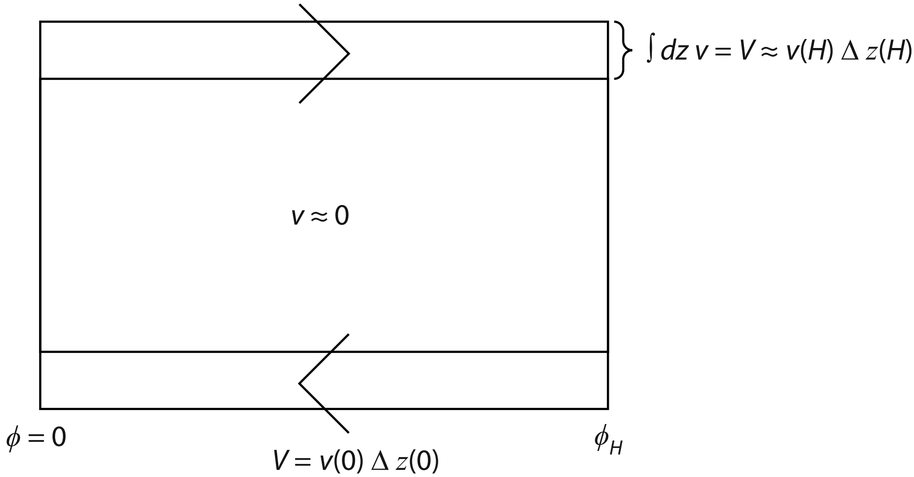

Fig. 9.14 Meridional flux in the Hadley cell is confined to thin layers near the top and bottom boundaries

These additional assumptions lead to the following estimate for the vertical-mean heat flux:

$$\frac{1}{\theta_0}\int_0^H dz v\theta \approx \frac{1}{\theta_0}\left[(V\theta)|_{z=H} + (V\theta)|_{z=0}\right] = V|_{z=H}\,\Delta_v \tag{9.109}$$

Likewise one obtains for the meridional momentum flux

$$\int_0^H dz\, uv \approx (Vu)|_{z=H} + (Vu)|_{z=0} \approx V|_{z=H}\, u_M \tag{9.110}$$

where the assumption (9.67) has been used in the last step, together with the result (9.71) for the latitude dependence of the zonal wind in the upper troposphere. Combined with (9.109) one finds the useful result

$$\int_0^H dz\, uv = \frac{u_M}{\Delta_v}\frac{1}{\theta_0}\int_0^H dz\, v\theta \tag{9.111}$$

Again using the boundary conditions (9.23) and (9.26) we now take the vertical mean of the angular-momentum conservation (9.46) and obtain

$$\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\int_0^H dz\, vm\right) = \left[K\frac{\partial m}{\partial z}\right]_0^H \tag{9.112}$$

Here we can simplify the vertical mean of the meridional angular-momentum flux on the left-hand side by inserting (9.45) and using the fact that the meridional mass flux vanishes (9.106). Hence

$$\int_0^H dz\, vm = a\cos\phi \int_0^H dz\, uv \tag{9.113}$$

On the right-hand side we use the fact that, due to the boundary conditions

$$K\frac{\partial m}{\partial z} = a\cos\phi K\frac{\partial u}{\partial z} = \begin{cases} 0 & z = H \\ a\cos\phi\, Cu & z = 0 \end{cases} \tag{9.114}$$

and hence

$$\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}\left(a\cos^2\phi\int_0^H dz\, uv\right) = -a\cos\phi\, C\, u|_{z=0} \tag{9.115}$$

Therefore the surface wind is

$$\begin{aligned} u|_{z=0} &= -\frac{1}{Ca\cos^2\phi}\frac{\partial}{\partial\phi}\left(\cos^2\phi\int_0^H dz\, uv\right) \\ &= -\frac{1}{\Delta_v Ca\cos^2\phi}\frac{\partial}{\partial\phi}\left(\cos^2\phi\frac{u_M}{\theta_0}\int_0^H dz\, v\theta\right) \end{aligned} \tag{9.116}$$

Finally using the results (9.102) and (9.71) yields

$$z = 0: \qquad u = -\frac{25}{18}\frac{\Omega a H \Delta_H}{C\tau\Delta_v}R^2\left[\left(\frac{\phi}{\phi_H}\right)^2 - \frac{10}{3}\left(\frac{\phi}{\phi_H}\right)^4 + \frac{7}{3}\left(\frac{\phi}{\phi_H}\right)^6\right] \tag{9.117}$$

In Fig. 9.15 the profile is shown together with numerical results for $K > 0$. The easterlies in the tropics are quite prominent.

9.2.5 The Summer–Winter Asymmetry

Held and Hou (1980) have assumed symmetry with respect to the equator. This is a reasonable assumption for spring and autumn, because in these seasons the radiative equilibrium is almost symmetric. In summer and winter, however, neither the radiative-equilibrium potential temperature nor the resulting Hadley circulation are symmetric. This situation is sketched in Fig. 9.16. On the summer and winter sides the cells are bounded by the latitudes ϕ_S and ϕ_W. The common rising branch of both cells is located at latitude ϕ_1. Following Lindzen

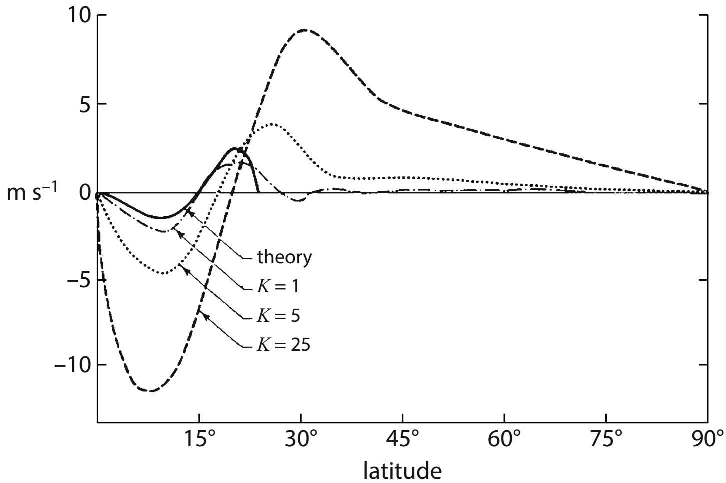

Fig. 9.15 Latitude dependence of the surface winds in the tropics. The analytical result for $K = 0$ is shown together with the numerical results for $K > 0$. Reprinted from Held and Hou (1980)

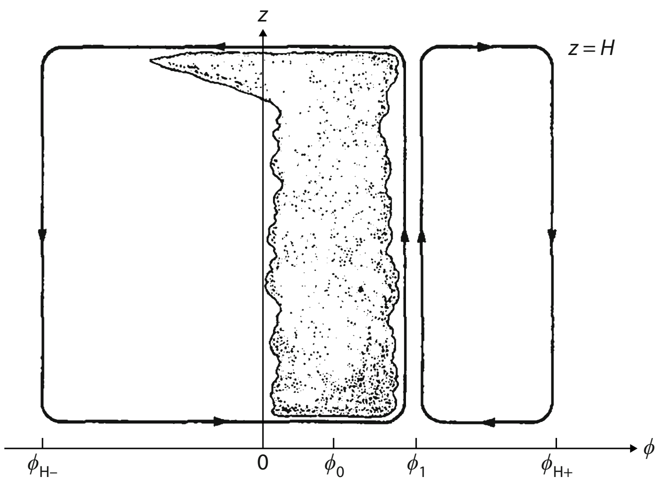

Fig. 9.16 Schematic representation of the summer–winter asymmetry of the Hadley circulation. On the summer and winter side the cells are bounded by the latitudes ϕ_S (here ϕ_{H+}) and ϕ_W (here ϕ_{H-}), respectively. The common rising branch of both cells is located at latitude ϕ_1. The latitude of the maximum of the radiative-equilibrium potential temperature is ϕ_0. Reprinted from Lindzen and Hou (1988)

and Hou (1988), the radiative-equilibrium potential temperature is approximated by

$$\frac{\theta_E}{\theta_0} = 1 + \frac{\Delta_H}{3} - \Delta_H\,(\sin\phi - \sin\phi_0)^2 + \Delta_v\left(\frac{z}{H} - \frac{1}{2}\right) \tag{9.118}$$

It reaches its maximum at all altitudes at the latitude ϕ_0. Note that in general $\phi_1 \neq \phi_0$. The latitude of the rising branch is to be determined from the calculations.

The assumptions for these are the same as the ones used in the symmetric calculations of Held and Hou (1980), assumption v) excepted. Thus we again assume conservation of angular-momentum in the upper branch of both cells. Hence

$$z = H: \qquad m = m|_{\phi=\phi_1} \tag{9.119}$$

Because

$$u(\phi_1, H) = 0 \tag{9.120}$$

this leads to

$$\Omega a^2\cos^2\phi + ua\cos\phi = \Omega a^2\cos^2\phi_1 \tag{9.121}$$

or

$$\boxed{z = H: \qquad u = u_M = \Omega a\frac{\cos^2\phi_1 - \cos^2\phi}{\cos\phi}} \tag{9.122}$$

One should note that the resulting winds at the equator are always easterlies.

For the calculation of the distribution of the potential temperature we assume generalized geostrophic balance, hydrostatics, and we neglect the zonal surface winds in comparison with their counterparts in the upper troposphere. One obtains

$$z = H: \qquad fu + \frac{u^2}{a}\tan\phi = -\frac{gH}{a\theta_0}\frac{\partial\{\theta\}}{\partial\phi} \tag{9.123}$$

Combined with (9.122), this leads to

$$\frac{\Omega^2 a}{2}\frac{\partial}{\partial\phi}\frac{\left(\sin^2\phi_1 - \sin^2\phi\right)^2}{\cos^2\phi} = -\frac{gH}{a\theta_0}\frac{\partial\{\theta\}}{\partial\phi} \tag{9.124}$$

Integration yields

$$\boxed{\frac{\{\theta\}}{\theta_0} = \frac{\{\theta\}(\phi_1)}{\theta_0} - \frac{\Omega^2 a^2}{2gH}\frac{\left(\sin^2\phi_1 - \sin^2\phi\right)^2}{\cos^2\phi}} \tag{9.125}$$

Finally the integration of the entropy equation without turbulent diffusion over the total area of either of the two cells yields

$$\int_{\phi_W}^{\phi_1} d\phi\, a(\{\theta\} - \{\theta_E\}) = 0 \tag{9.126}$$

$$\int_{\phi_1}^{\phi_S} d\phi\, a(\{\theta\} - \{\theta_E\}) = 0 \tag{9.127}$$

where Gauss' theorem has been applied. Furthermore, outside of the circulation cells one has the radiative-equilibrium solution. Continuity of potential temperature requires

$$\{\theta\}(\phi_S) = \{\theta_E\}(\phi_S) \tag{9.128}$$

$$\{\theta\}(\phi_W) = \{\theta_E\}(\phi_W) \tag{9.129}$$

Hence we have the four equations (9.126)–(9.129) for the four unknowns ϕ_S, ϕ_W, ϕ_1, and $\{\theta\}(\phi_1)$. They can only be solved numerically. Solutions for northern-hemisphere spring and summer are shown in Fig. 9.17. Note that not only there is an asymmetry in the potential temperature in the second case, but that there is also a significantly larger deviation from radiative balance on the winter side. This indicates a far stronger circulation on this side.

9.2.6 The Wave-Driven Hadley Circulation

The models considered so far have different drawbacks. The neglect of the evolution in time (beyond the seasonal cycle) as well as the neglect of the effects of turbulent viscosity cause errors. The most important correction, however, concerns the wave forcing of the tropical circulation, that we did not need to consider in the zonally symmetric models. In the discussion of this aspect we extend the equations by the wave forcing. For the sake of easier progress we then again apply appropriate approximations, so that the results to be derived complement the findings of the models of Schneider (1977), Held and Hou (1980) and Lindzen and Hou (1988) instead of generalizing them.

First we recall that the product of $a\cos\phi$ with the inviscid zonal-momentum equation in the Boussinesq approximation

$$\frac{\partial u}{\partial t} + \mathbf{v}\cdot\nabla u - fv + \frac{u^2}{a}\tan\phi = -\frac{1}{a\cos\phi}\frac{\partial P}{\partial\lambda} \tag{9.130}$$

yields the conservation equation for angular-momentum

$$\frac{\partial m}{\partial t} + \nabla\cdot(\mathbf{v}m) = -\frac{\partial P}{\partial\lambda} \tag{9.131}$$

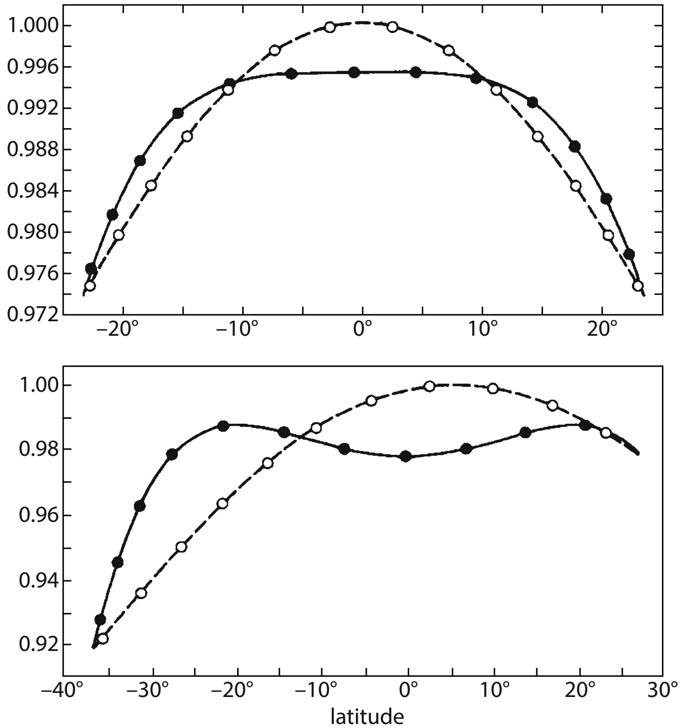

Fig. 9.17 Numerical results for the vertical-mean potential temperature $\{\theta\}$ (solid line) in northern-hemisphere spring (top picture, $\phi_0 = 0$) and in the summer (bottom picture, $\phi_0 = 6°$), together with the vertical mean $\{\theta_E\}$ (dashed) of the radiative-equilibrium potential temperature. Reprinted from Lindzen and Hou (1988)

Here we split all fields into zonal mean and waves

$$m = \langle m \rangle + m' \tag{9.132}$$

$$\mathbf{v} = \langle \mathbf{v} \rangle + \mathbf{v}' \tag{9.133}$$

$$P = \langle P \rangle + P' \tag{9.134}$$

and take the zonal mean of the equation. The result is

$$\frac{\partial \langle m \rangle}{\partial t} + \nabla \cdot (\langle \mathbf{v} \rangle \langle m \rangle) + \nabla \cdot \langle m' \mathbf{v}' \rangle = 0 \tag{9.135}$$

Averaging the continuity equation yields

$$\nabla \cdot \langle \mathbf{v} \rangle = 0 \tag{9.136}$$

so that one can write the angular-momentum equation

$$\frac{\partial \langle m \rangle}{\partial t} + \langle \mathbf{v} \rangle \cdot \nabla \langle m \rangle = -\nabla \cdot \langle m' \mathbf{v}' \rangle \tag{9.137}$$

or more in detail

$$\frac{\partial \langle m \rangle}{\partial t} + \frac{\langle v \rangle}{a} \frac{\partial \langle m \rangle}{\partial \phi} + \langle w \rangle \frac{\partial \langle m \rangle}{\partial z} = -\frac{1}{a \cos\phi} \frac{\partial}{\partial \phi} \left(\langle m' v' \rangle \cos\phi \right) - \frac{\partial}{\partial z} \langle m' w' \rangle \tag{9.138}$$

For estimates of the magnitude of each of the contributing terms we assume, for reasons of simplicity, synoptic scaling as in quasigeostrophic theory. First, due to (9.45) the advection terms on the left-hand side are

$$\langle w \rangle \frac{\partial \langle m \rangle}{\partial z} = a \cos\phi \, \langle w \rangle \frac{\partial \langle u \rangle}{\partial z} \tag{9.139}$$

$$\frac{\langle v \rangle}{a} \frac{\partial \langle m \rangle}{\partial \phi} = -a \cos\phi \, f \langle v \rangle + \frac{\langle v \rangle}{a} \frac{\partial}{\partial \phi} \left(a \cos\phi \, \langle u \rangle \right) \tag{9.140}$$

with $f = 2\Omega \sin\phi = \mathcal{O}(f_0)$. Here

$$\langle u \rangle = \mathcal{O}(U) \tag{9.141}$$

$$\langle v \rangle = \mathcal{O}(Ro\, U) \tag{9.142}$$

$$\langle w \rangle = \mathcal{O}\left(Ro \frac{H}{L} U \right) \tag{9.143}$$

whence

$$\langle w \rangle \frac{\partial \langle m \rangle}{\partial z} = \mathcal{O}\left(Ro\, a \frac{U^2}{L} \right) \tag{9.144}$$

$$a \cos\phi \, f \langle v \rangle = \mathcal{O}\left(a \frac{U^2}{L} \right) \tag{9.145}$$

$$\frac{\langle v \rangle}{a} \frac{\partial}{\partial \phi} \left(a \cos\phi \, \langle u \rangle \right) = \mathcal{O}\left(Ro\, a \frac{U^2}{L} \right) \tag{9.146}$$

so that one can neglect vertical advection in (9.138). Different than suggested by a comparison of (9.145) and (9.146), however, we keep the total merdional advection of angular-momentum, i.e., the term in (9.146) is not neglected! This is done on the one hand because, especially in the tropics but also in the subtropics, the dominance of the Coriolis term in (9.145) is less pronounced than in midlatitudes. On the other hand the term in (9.146) is the one responsible, in the zonally symmetric case, for the impact of angular-momentum conservation on the subtropical jet stream.

We finish the analysis of the angular-momentum Eq. (9.138) by considering the flux terms on the right-hand side. Due to (9.45) these are

$$\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\,\langle m'v'\rangle\right) = \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}\left(a\cos^2\phi\,\langle u'v'\rangle\right) \tag{9.147}$$

$$\frac{\partial}{\partial z}\langle m'w'\rangle = a\cos\phi\frac{\partial}{\partial z}\langle u'w'\rangle \tag{9.148}$$

Herein

$$u' = \mathcal{O}(U) \tag{9.149}$$

$$v' = \mathcal{O}(U) \tag{9.150}$$

$$w' = \mathcal{O}\left(Ro\frac{H}{L}U\right) \tag{9.151}$$

so that

$$\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\,\langle m'v'\rangle\right) = \mathcal{O}\left(a\frac{U^2}{L}\right) \tag{9.152}$$

$$\frac{\partial}{\partial z}\langle m'w'\rangle = \mathcal{O}\left(Ro\,a\frac{U^2}{L}\right) \tag{9.153}$$

Hence the vertical angular-momentum flux can be neglected in comparison with the horizontal flux.

In summary, a reasonable approximation of the angular-momentum equation is

$$\frac{\partial\langle m\rangle}{\partial t} + \frac{\langle v\rangle}{a}\frac{\partial\langle m\rangle}{\partial\phi} \approx -\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}\left(\langle u'v'\rangle a\cos^2\phi\right) \tag{9.154}$$

The momentum flux on the right-hand side is shown both for northern-hemispheric winter and summer in Fig. 9.18. Now consider, e.g., the upper branch of the Hadley cell in the northern hemisphere. There

$$\langle v\rangle > 0 \tag{9.155}$$

and also

$$-\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}\left(a\cos^2\phi\,\langle u'v'\rangle\right) < 0 \tag{9.156}$$

Hence, in the steady case

$$\frac{\partial\langle m\rangle}{\partial t} = 0 \tag{9.157}$$

angular momentum decreases with increasing latitude, i.e.,

$$\frac{\partial\langle m\rangle}{\partial\phi} < 0 \tag{9.158}$$

As compared to the zonally symmetric case, *the jet stream is weakened by the wave forcing*. Analogous findings hold for the southern hemisphere. A corresponding result from a simplified climate model is shown in Fig. 9.19.

Next we consider the entropy equation

$$\frac{\partial\theta}{\partial t} + \nabla\cdot(\mathbf{v}\theta) = -\frac{\theta-\theta_E}{\tau} \tag{9.159}$$

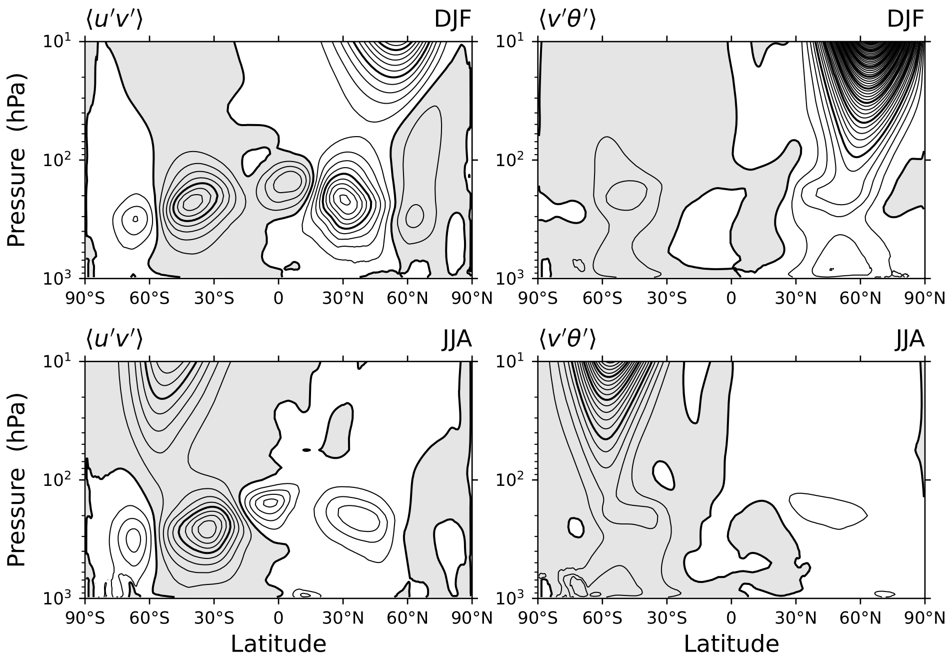

Fig. 9.18 Climatology of the meridional fluxes of potential temperature (right column, contour interval 10K m/s) and zonal-momentum (left, contour interval $10\,\mathrm{m^2/s^2}$) in northern-hemispheric winter (top row) and summer (bottom). Shaded regions indicate negative values. Data from ERA5 (Hersbach et al., 2020)

without turbulent diffusion. Similar to the analysis of the angular-momentum equation one can rewrite and simplify it as

$$
\begin{aligned}
\frac{\partial \langle\theta\rangle}{\partial t} + \nabla\cdot(\langle\mathbf{v}\rangle\langle\theta\rangle) &= -\nabla\cdot\langle\mathbf{v}'\theta'\rangle - \frac{\langle\theta\rangle-\theta_E}{\tau}\\
&= -\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\,\langle v'\theta'\rangle\right) - \frac{\partial}{\partial z}\langle w'\theta'\rangle - \frac{\langle\theta\rangle-\theta_E}{\tau}\\
&\approx -\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\,\langle v'\theta'\rangle\right) - \frac{\langle\theta\rangle-\theta_E}{\tau} \qquad (9.160)
\end{aligned}
$$

or even

$$
\frac{\partial \langle\theta\rangle}{\partial t} + \nabla\cdot(\langle\mathbf{v}\rangle\langle\theta\rangle) = -\frac{\langle\theta\rangle-\theta_E^W}{\tau} \qquad (9.161)
$$

with a wave-modified equilibrium potential temperature

$$
\theta_E^W = \theta_E - \frac{\tau}{a\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\,\langle v'\theta'\rangle\right) \qquad (9.162)
$$

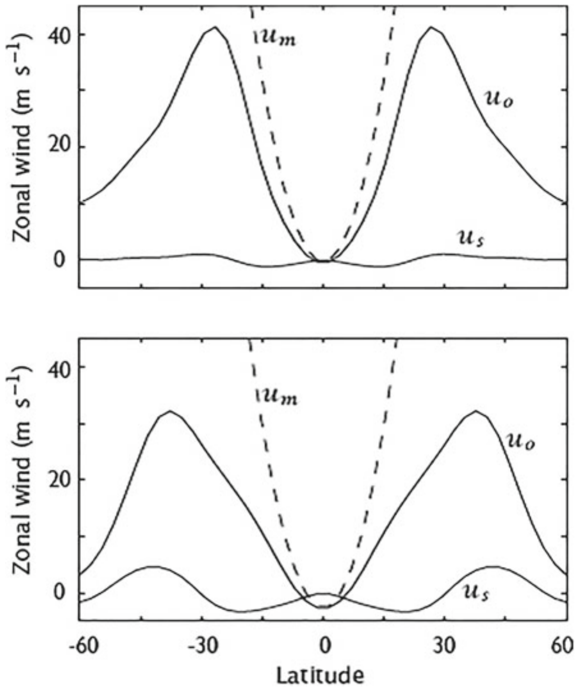

Fig. 9.19 Upper-tropospheric zonal-mean zonal wind from simulations with a general circulation model without waves (top panel) and with waves (bottom). Here u_0 is the wind in the upper tropopshere, and u_s the surface wind. The wind u_M due to angular-momentum conservation, following Held and Hou (1980) is shown as well. Reprinted from Vallis (2006) with permission from Cambridge University Press

In the tropics of the northern hemisphere one has, however, in northern-hemisphere winter, as shown in Fig. 9.18,

$$\frac{\partial^2}{\partial\phi^2}\langle v'\theta'\rangle > 0 \tag{9.163}$$

so that the waves enhance the latitudinal gradient of the equilibrium potential temperature, i.e.

$$\frac{\partial\theta_E^W}{\partial\phi} < \frac{\partial\theta_E}{\partial\phi} < 0 \tag{9.164}$$

Hence, the *meridional circulation of the Hadley cell is enhanced by the waves.* Analogous findings hold for the southern hemisphere. Corresponding simulation results are shown in Fig. 9.20.

This can be understood further via another systematic perspective that will also be of use below in the discussion of the Ferrel cells. We reconsider the angular-momentum equation (9.154). Therein one has, due to (9.45),

$$\frac{\partial\langle m\rangle}{\partial t} = a\cos\phi\frac{\partial\langle u\rangle}{\partial t} \tag{9.165}$$

The meridional angular-momentum advection can be decomposed following (9.140), with the corresponding order-of-magnitude estimates (9.145) and (9.146). Different than done above, however, we do not keep the term in (9.146) because we want to focus on the wave forcing. We therefore also approximate $f \approx f_0$ and obtain thus from the angular-momentum

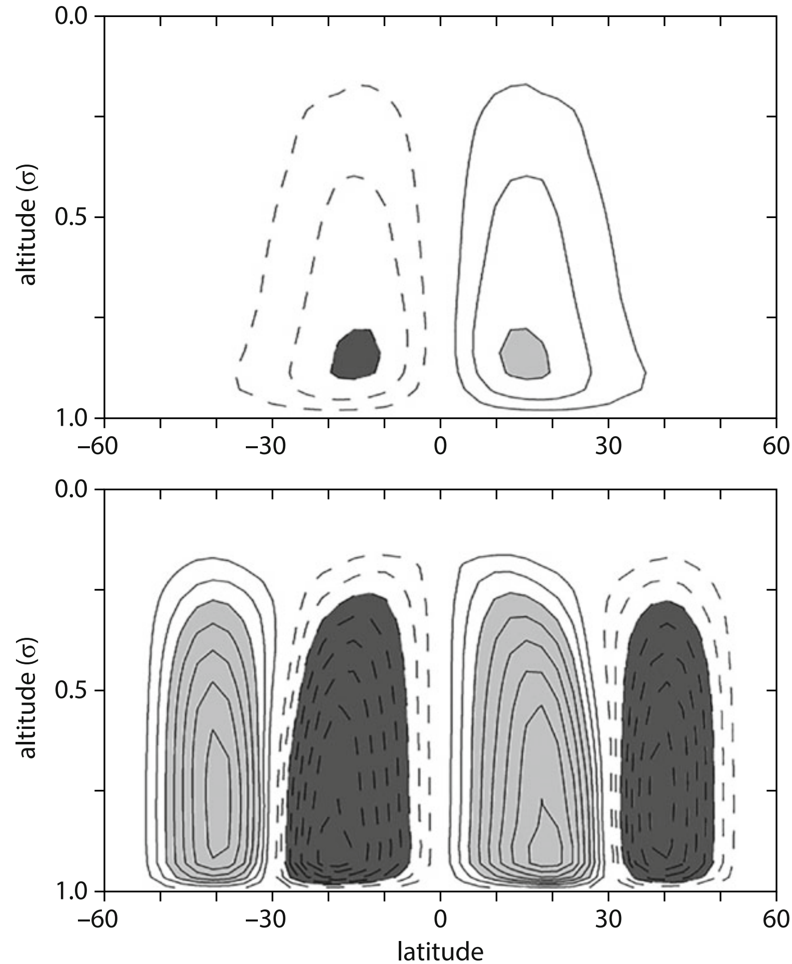

Fig. 9.20 As Fig. 9.19, but now for the meridional streamfunction. Reprinted from Vallis (2006) with permission from Cambridge University Press

equation

$$\frac{\partial \langle u \rangle}{\partial t} - f_0 \langle v \rangle \approx M \tag{9.166}$$

with

$$M = -\frac{1}{a^2 \cos^2 \phi} \frac{\partial}{\partial \phi} \left(\langle u' v' \rangle a \cos^2 \phi \right) \tag{9.167}$$

Likewise we simplify the mean entropy Eq. (9.160). Therein

$$\theta = \overline{\theta} + \tilde{\theta} \tag{9.168}$$

and hence, also using the non-divergence of the velocity field,

$$\nabla \cdot (\langle \mathbf{v} \rangle \langle \theta \rangle) = \langle \mathbf{v} \rangle \cdot \nabla \langle \theta \rangle = \frac{\langle v \rangle}{a} \frac{\partial \langle \tilde{\theta} \rangle}{\partial \phi} + \langle w \rangle \frac{d\overline{\theta}}{dz} + \langle w \rangle \frac{\partial \langle \tilde{\theta} \rangle}{\partial z} \tag{9.169}$$

According to synoptic scaling in quasigeostrophic theory

$$\tilde{\theta} = \mathcal{O}\left(Ro^2 \theta_0\right) \tag{9.170}$$

$$\frac{d\overline{\theta}}{dz} = \mathcal{O}\left(Ro \frac{\theta_0}{H}\right) \tag{9.171}$$

and the magnitudes of the zonal-mean meridional and vertical winds can be estimated via (9.142) and (9.143). Therefore

$$\nabla \cdot (\langle \mathbf{v} \rangle \langle \theta \rangle) \approx \langle w \rangle \frac{d\overline{\theta}}{dz} \tag{9.172}$$

The product of (9.160) with g/θ_0 hence yields

$$\frac{\partial \langle b \rangle}{\partial t} + \langle w \rangle N^2 \approx J \tag{9.173}$$

with

$$J = -\frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left(\langle b' v' \rangle \cos \phi\right) - \frac{g}{\theta_0} \frac{\langle \theta \rangle - \theta_E}{\tau} \tag{9.174}$$

and

$$N^2 = \frac{g}{\theta_0} \frac{d\overline{\theta}}{dz} \tag{9.175}$$

For reasons of simplicity we henceforth use Cartesian coordinates. Then the mean continuity equation is

$$\frac{\partial \langle v \rangle}{\partial y} + \frac{\partial \langle w \rangle}{\partial z} = 0 \tag{9.176}$$

Hence there is a streamfunction ψ so that the mean-flow fields are

$$\langle v \rangle = -\frac{\partial \psi}{\partial z} \tag{9.177}$$

$$\langle w \rangle = \frac{\partial \psi}{\partial y} \tag{9.178}$$

Moreover, within quasigeostrophic theory one has the thermal-wind relation

$$f_0 \frac{\partial \langle u \rangle}{\partial z} = -\frac{\partial \langle b \rangle}{\partial y} \tag{9.179}$$

Using all this in $f_0 \partial(9.166)\,/\partial z + \partial(9.173)/\partial y$ finally yields the important relationship

$$\boxed{f_0^2 \frac{\partial^2 \psi}{\partial z^2} + N^2 \frac{\partial^2 \psi}{\partial y^2} = f_0 \frac{\partial M}{\partial z} + \frac{\partial J}{\partial y}} \tag{9.180}$$

This is an elliptic equation for the streamfunction, holding even in the case of time-dependent fields, but not containing any time derivatives! The mean circulation can always be determined directly from the momentum forcing M and the heating J, both including contributions from the wave forcing.

The qualitative handling of the equation is quite intuitive. Obviously, as far as the sign is concerned,

$$f_0^2 \frac{\partial^2 \psi}{\partial z^2} + N^2 \frac{\partial^2 \psi}{\partial y^2} \sim -\psi \tag{9.181}$$

because maxima (minima) are associated with negative (positive) curvature. This helps in analyzing the impact of heat and momentum fluxes.

- One can see in Fig. 9.18 that even without direct heating

$$\frac{\partial J}{\partial y} < 0 \tag{9.182}$$

  because the heat flux has a minimum in the deep tropics so that $\partial^2 \langle v' b' \rangle / \partial y^2 > 0$. This leads to a positive streamfunction $\psi > 0$ with rising air masses in the tropics and sinking air masses in the subtropics. This is summarized in Fig. 9.21. In the southern hemisphere one has likewise $\psi < 0$.
- Likewise one derives from Fig. 9.18 that in the tropopshere

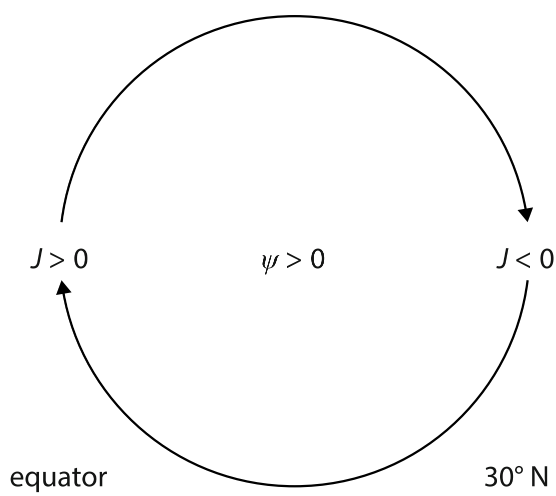

Fig. 9.21 Qualitative impact of the heat fluxes on the Hadley circulation. They cause relative heating in the tropics and cooling in the subtropics, so that air masses rise in the tropics and sink in the extratropics

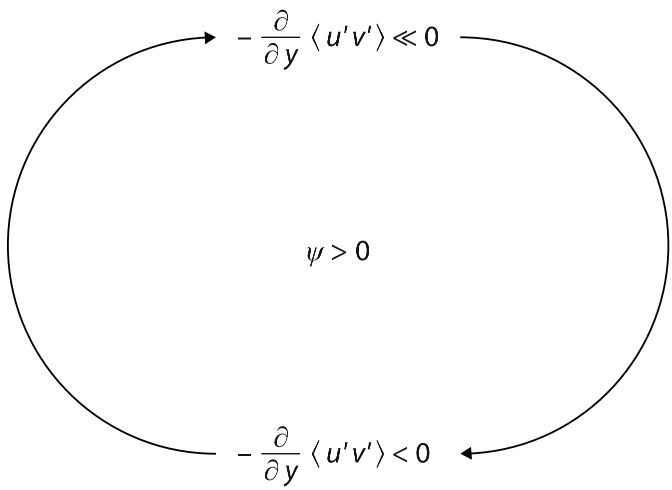

Fig. 9.22 Impact of the momentum fluxes on the Hadley circulation. As the heat fluxes they enhance the direct Hadley circulation

$$\frac{\partial M}{\partial z} < 0 \tag{9.183}$$

because the divergence of the momentum flux ($\partial \langle u'v' \rangle / \partial y > 0$) increases with altitude. This also leads to $\psi > 0$, as summarized in Fig. 9.22. Again one obtains for the southern hemisphere $\psi < 0$.

- One also sees that a reduction of the static stability N^2 leads to an enhancement of the meridional circulation as well.

9.2.7 Summary

To *leading order* the Hadley circulation in the tropics *can already be understood without the impact of eddies.* Those, however, represent an important correction.

- A discussion of the dynamics without waves can be done using the *stationary zonally symmetric primitive equations* in Boussinesq approximation. Essential are *radiative heating,* with a potential temperature characterizing radiative equilibrium that decreases in spring and fall from the equator to the poles, and momentum exchange with the solid earth via *turbulent surface friction*. In the inviscid case the equations admit a geostrophic-hydrostatic equilibrium solution without meridional circulation. Even the weakest friction, however, makes this solution impossible, and a solution is to be found where a *maximum of angular momentum at the ground is accompanied by easterlies* in its vicinity.

- In the case of weak turbulent viscosity an analytical solution is possible with a *subtropical jet stream resulting from angular-momentum conservation in a merdidional circulation directed polewards in the upper troposphere.* The latitude dependence of the corresponding potential temperature follows from the thermal wind and a continuous transition to *radiative equilibrium outside of the Hadley cell*. One finds that the *horizontal extent of the Hadley circulation* increases with the equator-pole contrast in radiative-equilibrium potential temperature, while it decreases with radius and rotation of the planet. *The stronger the equator-pole contrast the stronger the circulation,* and the stronger as well the reduction of potential temperature in the tropics. The corresponding heat flux is everywhere directed from the tropics to the middle latitudes. An *equilibrium between angular-momentum flux and the viscous-turbulent angular-momentum sink by surface friction* leads in the vertically integrated angular-momentum equation to *surface easterlies near the equator*.
- A modification of the zonally symmetric model for *summer and winter conditions* leads to easterlies everywhere above the equator. The *Hadley cell on the winter side is considerably stronger* than the summer cell.
- *Eddies turn out to have a considerable influence.* With the latitude dependence of their momentum fluxes in the upper troposphere they represent an angular-momentum sink so that the *subtropical jet stream with eddies is weaker than without eddies*. In addition, the latitude dependence of the eddy heat fluxes *enhances the equator-pole contrast between heating and cooling so that the Hadley circulation is forced additionally.* These impacts can be captured quite conveniently in an elliptic equation for the mass streamfunction. They are the stronger the weaker the stratification is.

9.3 The Circulation in the Midlatitudes

While the circulation in the tropics can be described to some extent without waves, the midlatitudes are not to be understood without the effect of waves at all. The dependency of the heating rates on longitude, for example because of the land–sea contrast, orographic wave generation and particularly baroclinic instability continuously excite waves in the extra tropics. Hence the dynamics of this latitude region is intrinsically turbulent. Important characteristics of the resulting circulation are the Ferrel cells and the barotropic jet stream in the midlatitudes. In the following section we want to discuss the dynamics of these phenomena.

9.3.1 The Phenomenology of the Ferrel Cell

For a phenomenological description of the Ferrel cells we can go back to the considerations of the last section. In the context of Boussinesq theory it is possible here as well to introduce a meridional streamfunction ψ defining the zonal-mean meridional and vertical winds via (9.177) and (9.178). Using the effective heating rate

$$J = -\frac{\partial}{\partial y}\langle b'v'\rangle - \frac{g}{\theta_0}\frac{\langle\theta\rangle - \theta_E}{\tau} \tag{9.184}$$

and the effective acceleration

$$M = -\frac{\partial}{\partial y}\langle u'v'\rangle + \frac{\partial}{\partial z}\left(K\frac{\partial\langle u\rangle}{\partial z}\right) \tag{9.185}$$

supplemented here by the effect of turbulent friction in the boundary layer, this streamfunction can be determined from the elliptical Eq. (9.180). The corresponding derivation has not made use of any specific aspects of tropical dynamics, so that all can be used in midlatitudes right away. In fact the application of quasigeostrophic theory is much better justified in the present context. Different here is that we can neglect the direct heating, as we did in the tropics with regard to turbulent friction outside of the boundary layer, and the sign of the wave forcing changes as well.

Thus, according to Fig. 9.18 one finds that the momentum-flux convergence in midlatitudes

$$\frac{\partial}{\partial y}\langle u'v'\rangle < 0 \tag{9.186}$$

increases with altitude up to the tropopause so that

$$\frac{\partial M}{\partial z} = -\frac{\partial^2}{\partial z\partial y}\langle u'v'\rangle > 0 \tag{9.187}$$

As a consequence of the baroclinic instability active there the heat flux has its maximum in midlatitudes so that

$$\frac{\partial^2}{\partial y^2}\langle v'b'\rangle < 0 \tag{9.188}$$

whence

$$\frac{\partial J}{\partial y} > 0 \tag{9.189}$$

All of this taken together shows that the streamfunction

$$\psi < 0 \tag{9.190}$$

must be negative, so that the corresponding circulation is opposite to the one of the Hadley cell. In the southern hemisphere the signs reverse correspondingly.

Alternatively the circulation outside of the boundary layer can also be derived directly from the momentum Eq. (9.166) and the buoyancy Eq. (9.173). In the climatological mean all time derivatives vanish in winter and summer, and one obtains with (9.184) and (9.185), neglecting temperature relaxation and turbulent viscosity,

$$\langle v \rangle = \frac{1}{f_0} \frac{\partial}{\partial y} \langle u'v' \rangle \tag{9.191}$$

$$\langle w \rangle = -\frac{1}{N^2} \frac{\partial}{\partial y} \langle v'b' \rangle \tag{9.192}$$

where the zonal means also indicate time means. Following Fig. 9.18 one has in midlatitudes $\partial \langle u'v' \rangle / \partial y < 0$. This explains the sign of $\langle v \rangle$ in the upper part of the Ferrel cells. One sees from this figure as well that the sign of $\langle w \rangle$ at the meridional boundaries of the Ferrel cell can be explained directly by the sign of $\partial \langle v'b' \rangle / \partial y$.

The surface winds in midlatitudes can be understood by two lines of argmentation that do not differ in essence, however. Consider the climatological mean of the zonal-mean zonal-momentum equation (9.166). The acceleration according to (9.185) includes the effect of the turbulent friction in the boundary layer, and one obtains

$$-f_0 \langle v \rangle = -\frac{\partial}{\partial y} \langle u'v' \rangle + \frac{\partial}{\partial z} \left(K \frac{\partial \langle u \rangle}{\partial z} \right) \tag{9.193}$$

- On the one hand, one can integrate this equation vertically over the boundary layer with the thickness Δz. Because the horizontal momentum fluxes are negligible there, one finds

$$-f_0 V \approx \left[K \frac{\partial \langle u \rangle}{\partial z} \right]_0^{\Delta z} \tag{9.194}$$

  with

$$V = \int_0^{\Delta z} dz \langle v \rangle > 0 \tag{9.195}$$

  the vertical integral of the meridional wind. Here, the boundary conditions are such that the turbulent momentum flux vanishes at the top of the boundary layer. At the lower boundary the turbulent momentum flux can be approximated by a drag coefficient,

$$K \frac{\partial \langle u \rangle}{\partial z} = \begin{cases} 0 & \text{at } z = \Delta z \\ C \langle u \rangle & \text{at } z = 0 \end{cases} \tag{9.196}$$

  Hence one obtains

$$z = 0 : \langle u \rangle = \frac{f_0}{C} V > 0 \tag{9.197}$$

  i.e., westerlies prevail so that friction is balanced by the Coriolis effect.

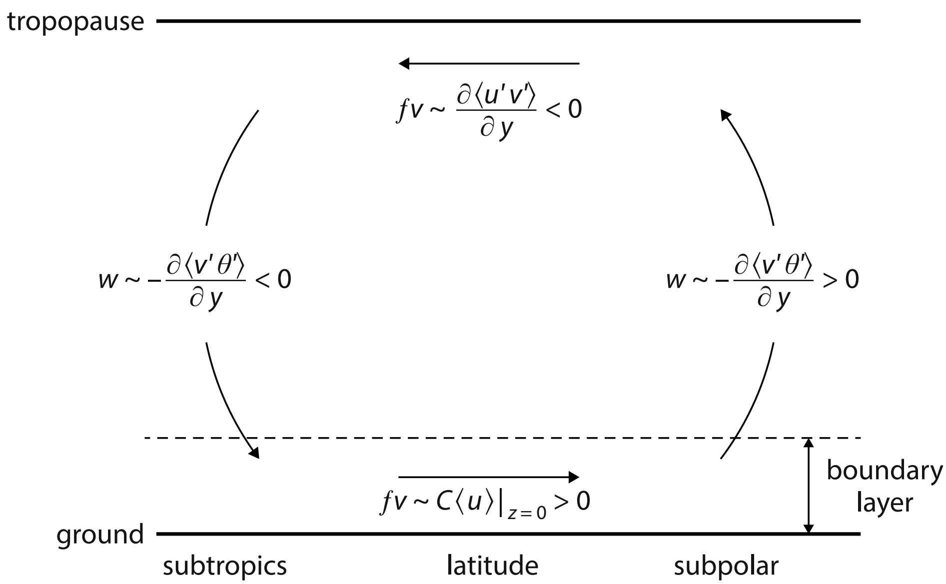

Fig. 9.23 Wave forcing of the Ferrel cell and the surface westerlies in midlatitudes

- On the other hand, one can integrate over the whole troposphere, yielding

$$- f_0 \int_0^H dz \langle v \rangle = - \int_0^H dz \frac{\partial}{\partial y} \langle u'v' \rangle - C \ \langle u \rangle|_{z=0} \tag{9.198}$$

  But according to (9.106) the total meridional mass flux vanishes, simplifying the equation to

$$z = 0: \qquad \langle u \rangle = -\frac{1}{C} \int_0^H dz \frac{\partial}{\partial y} \langle u'v' \rangle > 0 \tag{9.199}$$

  because the mid latitude momentum flux is convergent, as can also be read from Fig. 9.18.

Following Vallis (2006), the whole is summarized in Fig. 9.23.

9.3.2 Eddy Fluxes and Barotropic Jet Stream

A distinguished characteristic of the midlatitudes is their barotropic jet stream, most evident in local cuts without zonal average. In Fig. 9.24 the baroclinic jet stream in the subtropics, with surface easterlies, is clearly distinguishable from the barotropic jet stream in midlatitudes, where westerlies extend to the ground. This wave-driven phenomenon shall be discussed

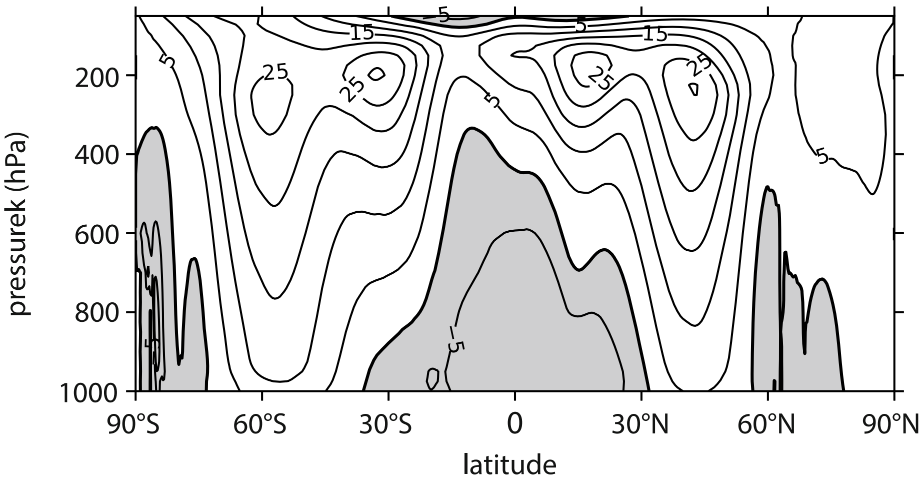

Fig. 9.24 Latitude–altitude dependence of the zonal wind at 150°W above the Pacific in northern-hemispheric spring. Data from ERA5 (Hersbach et al., 2020)

in the following. We will also see there that the configuration of the eddy fluxes causing this jet stream are a direct result of baroclinic instability and the conservation of vorticity as described by the Kelvin theorem.

The Basic Mechanism

Baroclinicity is essential in the dynamics of the jet stream, this however only for the explanation of the mid latitude wave source by baroclinic instability. Beyond this all can be discussed within the framework of barotropic dynamics. Take, e.g., a barotropic incompressible β-channel with periodic boundary conditions in zonal direction. Because the flow is purely horizontal one has from incompressibility

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9.200}$$

therefrom

$$\frac{\partial \langle v \rangle}{\partial y} = 0 \tag{9.201}$$

and hence also

$$\langle v \rangle = 0 \tag{9.202}$$

because the flow at the meridional boundaries of the channel must be zonal. Moreover, there is only vertical relative vorticity

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \tag{9.203}$$

The planetary vorticity is $f = f_0 + \beta y$, and the absolute vorticity $\omega_{az} = f + \zeta$ consists of the latter and relative vorticity. Assume moreover that the atmosphere is initially at rest so that its absolute vorticity is identical with the planetary vorticity, increasing from south to north. Now the atmosphere is put in midlatitudes into irregular motion, e.g., by a baroclinic wave source. Following the Kelvin theorem (4.33), outside of the region of the wave source all material surface elements conserve their absolute vorticity flux

$$\Gamma_a = \omega_{az} dS \tag{9.204}$$

The velocity field, however, is non-divergent, and hence due to the two-dimensional variant of (1.12) the material surface dS is conserved as well, so that the material surface elements transport their absolute vorticity. The latter is given initially by the planetary vorticity so that surface elements moving northwards carry low absolute vorticity and southwards moving surface elements transport high vorticity. Hence, in the zonal mean one obtains *outside of the baroclinic source region* a negative vorticity flux

$$\langle v\omega_{az}\rangle = \langle v'\omega'_{az}\rangle < 0 \tag{9.205}$$

From the definition of the absolute vorticity it is obvious that the zonal-mean flux of absolute vorticity agrees with that of relative vorticity, i.e.,

$$\langle v'\omega'_{az}\rangle = \langle v'\zeta'\rangle \tag{9.206}$$

This relative-vorticity flux is, due to (9.200),

$$\begin{aligned} v'\zeta' = v'\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right) &= \frac{\partial}{\partial x}\frac{v'^2}{2} - \frac{\partial}{\partial y}(u'v') + u'\frac{\partial v'}{\partial y} = \frac{\partial}{\partial x}\frac{v'^2}{2} - \frac{\partial}{\partial y}(u'v') - u'\frac{\partial u'}{\partial x} \\ &= \frac{\partial}{\partial x}\left(\frac{v'^2}{2} - \frac{u'^2}{2}\right) - \frac{\partial}{\partial y}(u'v') \end{aligned} \tag{9.207}$$

so that its zonal mean agrees with the zonal-mean momentum-flux convergence

$$\langle v'\zeta'\rangle = -\frac{\partial}{\partial y}\langle u'v'\rangle = -\frac{\partial}{\partial y}\langle uv\rangle \tag{9.208}$$

Because the flow at the meridional boundaries of the channel is zonal, this implies that the meridional average of $\langle v'\zeta'\rangle$ vanishes, as well as that of $\langle v'\omega'_{az}\rangle$. Therefore one has *in the baroclinic source region*

$$\frac{\partial}{\partial y}\langle u'v'\rangle = -\langle v'\zeta'\rangle = -\langle v'\omega'_{az}\rangle < 0 \tag{9.209}$$

so that positive zonal-momentum is transported from the subtropics and polar regions into middle latitudes. Hence a jet stream with $\langle u\rangle > 0$ is caused in midlatitudes, flanked by easterlies. This is sketched in Fig. 9.25.

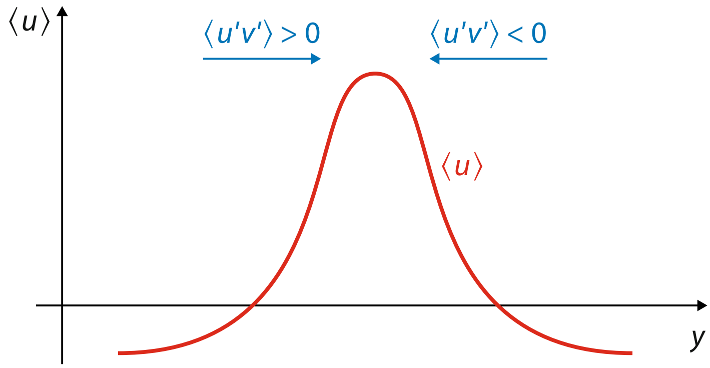

Fig. 9.25 Westerly jet stream with its easterly flanks, as it originates in midlatitudes from momentum-flux convergence, that is again due to a baroclinic source there and vorticity conservation according to the Kelvin theorem

A further angle to this result is offered by the properties of Rossby waves, now without the assumption of barotropicity and incompressibility, however within the framework of linear quaisgeostrophic theory. Consider the baroclinic instability as a mid latitude source of Rossby waves. Following (8.146) these propagate with a meridional group velocity

$$c_{gy} = \frac{\partial \langle \pi \rangle}{\partial y} \frac{2lk}{\left(k^2 + l^2 + \frac{f_0^2}{N^2} m^2 + \frac{1}{4L_{di}^2}\right)^2} \tag{9.210}$$

where k and l are the zonal and meridional wavenumber, respectively, and $\langle \pi \rangle$ the zonal mean of potential vorticity. Although the latter also contains contributions from the atmospheric flow field, its meridional derivative is dominated by that of the planetary vorticity, i.e.

$$\frac{\partial \langle \pi \rangle}{\partial y} \approx \beta > 0 \tag{9.211}$$

Because the waves originate from the mid latitude source region one finds that

$$\text{to the north of the source region:} \quad c_{gy} > 0 \Rightarrow kl > 0 \tag{9.212}$$

$$\text{to the south of the source region:} \quad c_{gy} < 0 \Rightarrow kl < 0 \tag{9.213}$$

This has consequences for the meridional momentum flux. The contribution of each Rossby wave is, due to (8.144),

$$\langle u'v' \rangle = -\frac{|A|^2}{\overline{\rho}} kl \tag{9.214}$$

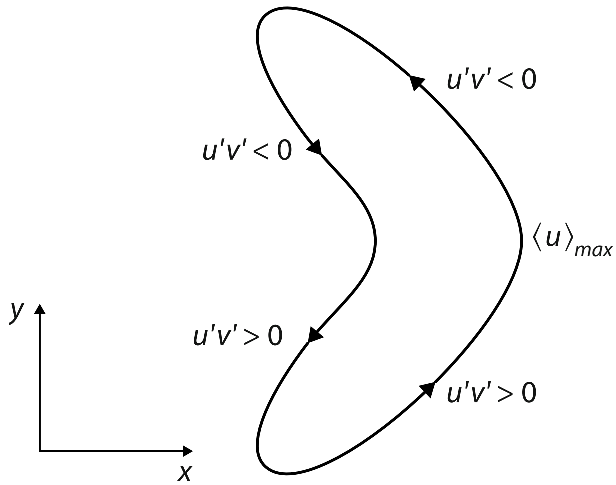

Fig. 9.26 Characteristic boomerang shape of the streamlines of midlatitudes Rossby waves with a convergent momentum flux

Hence

$$\text{to the north of the source region:} \quad \langle u'v' \rangle < 0 \tag{9.215}$$

$$\text{to the south of the source region:} \quad \langle u'v' \rangle > 0 \tag{9.216}$$

whence follows (9.209). The Rossby-wave streamlines have a characteristic boomerang shape as sketched in Fig. 9.26.

A Closed Description with Wave Source and Dissipative Sink

For the development of simple closed equations for the jet stream we return to barotropic dynamics. Consider a β-channel with constant density ρ_0 and purely horizontal flow. The equation of continuity then leads to (9.200). The horizontal-momentum equations are in this framework

$$\frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla u - f v = -\frac{\partial P}{\partial x} + F_u - D_u \tag{9.217}$$

$$\frac{\partial v}{\partial t} + \mathbf{v} \cdot \nabla v + f u = -\frac{\partial P}{\partial y} + F_v - D_v \tag{9.218}$$

where $P = p/\rho_0$ is the density-normalized pressure, $\mathbf{F}$ the baroclinic source, and $\mathbf{D}$ the viscous-turbulent term. As usual $\partial(9.218)/\partial x - \partial(9.217)/\partial y$ with (9.200) leads to the prognostic equation

$$\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta + \beta v = F_\zeta - D_\zeta \tag{9.219}$$

for relative vorticity (9.203), where

$$F_\zeta = \frac{\partial F_v}{\partial x} - \frac{\partial F_u}{\partial y} \tag{9.220}$$

is the baroclinic vorticity source, and

$$D_\zeta = \frac{\partial D_v}{\partial x} - \frac{\partial D_u}{\partial y} \tag{9.221}$$

the viscous-turbulent sink. Due to the non-divergence (9.200) the zonal-momentum equation (9.217) can also be written

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{u}u) - fv = -\frac{\partial P}{\partial x} + F_u - D_u \tag{9.222}$$

whence in the zonal mean

$$\frac{\partial \langle u \rangle}{\partial t} + \frac{\partial}{\partial y}\langle uv \rangle - f\langle v \rangle = \langle F_u \rangle - \langle D_u \rangle \tag{9.223}$$

Now, however, due to the non-divergence (9.200) and the impermeability of the meridional boundaries of the β-channel, the zonal-mean meridional velocity vanishes as in (9.202) so that

$$\langle uv \rangle = \langle u \rangle \langle v \rangle + \langle u'v' \rangle = \langle u'v' \rangle \tag{9.224}$$

Beyond this it is reasonable to assume

$$\langle F_u \rangle = 0 \tag{9.225}$$

because **F** represents the effect of baroclinic instability, generating essentially waves without zonal mean. Finally, we choose as most simple ansatz for the description of turbulent friction

$$\langle D_u \rangle = r\langle u \rangle \tag{9.226}$$

All of this taken together we obtain from (9.223)

$$\frac{\partial \langle u \rangle}{\partial t} = -\frac{\partial}{\partial y}\langle u'v' \rangle - r\langle u \rangle \tag{9.227}$$

But due to (9.208) the vorticity flux is identical with the momentum-flux convergence so that these equations can also be written, again using (9.202),

$$\frac{\partial \langle u \rangle}{\partial t} = \langle v'\zeta' \rangle - r\langle u \rangle \tag{9.228}$$

The mean-flow acceleration can be related to wave transience. The zonal mean of the vorticity equation (9.219) is

$$\frac{\partial \langle \zeta \rangle}{\partial t} + \frac{\partial}{\partial y}\langle v'\zeta' \rangle = \langle F_\zeta \rangle - \langle D_\zeta \rangle \tag{9.229}$$

Subtracting this from (9.219) results, under neglect of all terms nonlinear in the wave contributions, in a prognostic equation for the eddy vorticity,

$$\frac{\partial \zeta'}{\partial t} + \langle u \rangle \frac{\partial \zeta'}{\partial x} + \gamma v' = F'_\zeta - D'_\zeta \tag{9.230}$$

where

$$\gamma = \beta + \frac{\partial \langle \zeta \rangle}{\partial y} \tag{9.231}$$

is the meridional derivative of absolute vorticity. Similar to Sect. 8.3.2 we now assume that the latter varies only very slowly in time. Then, multiplication of the eddy-vorticity equation by ζ'/γ, and zonal averaging of the result, yields the equation

$$\frac{\partial \mathcal{A}}{\partial t} + \langle v'\zeta' \rangle = \frac{1}{\gamma} \left(\langle \zeta' F'_\zeta \rangle - \langle \zeta' D'_\zeta \rangle \right) \tag{9.232}$$

for the wave-action density

$$\mathcal{A} = \left\langle \frac{\zeta'^2}{2\gamma} \right\rangle \tag{9.233}$$

The sum of (9.228) and (9.232) is

$$\frac{\partial \langle u \rangle}{\partial t} + \frac{\partial \mathcal{A}}{\partial t} = \frac{1}{\gamma} \left(\langle \zeta' F'_\zeta \rangle - \langle \zeta' D'_\zeta \rangle \right) - r \langle u \rangle \tag{9.234}$$

This yields a barotropic variant of the non-accelaration theorem. In the time mean, here of interest, we find

$$r \langle u \rangle = \frac{1}{\gamma} \left(\langle \zeta' F'_\zeta \rangle - \langle \zeta' D'_\zeta \rangle \right) \tag{9.235}$$

Hence the mean zonal wind results from a balance between the baroclinic source $\langle \zeta' F'_\zeta \rangle$ of wave action, largest in midlatitudes, and the viscous-turbulent sink $\langle \zeta' D'_\zeta \rangle$. This balance integrates to zero: The meridional integral of (9.232) is in the time mean

$$0 = \frac{1}{\gamma} \int dy \left(\langle \zeta' F'_\zeta \rangle - \langle \zeta' D'_\zeta \rangle \right) \tag{9.236}$$

because, due to the meridional boundary conditions of the β-channel

$$\int dy \langle v'\zeta' \rangle = - \int dy \frac{\partial}{\partial y} \langle u'v' \rangle = 0 \tag{9.237}$$

It is obvious that in the middle latitudes, close to the baroclinic source,

$$\langle \zeta' F'_\zeta \rangle > \langle \zeta' D'_\zeta \rangle \tag{9.238}$$

whence

$$\langle u \rangle > 0 \tag{9.239}$$

while at the flanks of the jet stream

$$\langle \zeta' F'_\zeta \rangle < \langle \zeta' D'_\zeta \rangle \tag{9.240}$$

and hence

$$\langle u \rangle < 0 \tag{9.241}$$

The decisive balances are sketched in Figs. 9.27 and 9.28.

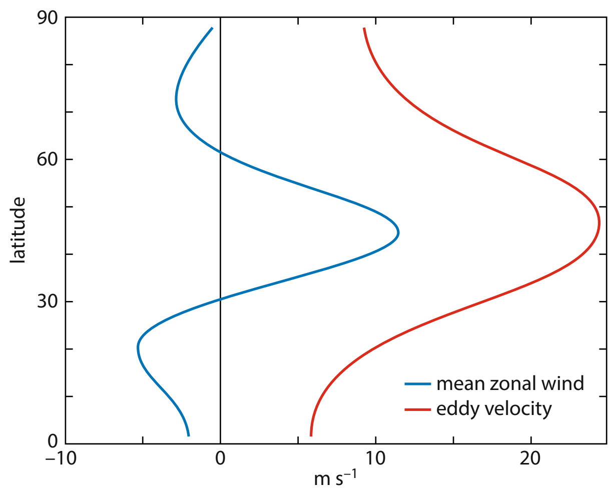

Fig. 9.27 Latitude distribution of the zonal-mean wind and the eddy velocity corresponding to the wave-action density, resulting from a barotropic model with mid latitude wave source and viscous-turbulent sink. Redrawn from Vallis (2006) with permission from Cambridge University Press

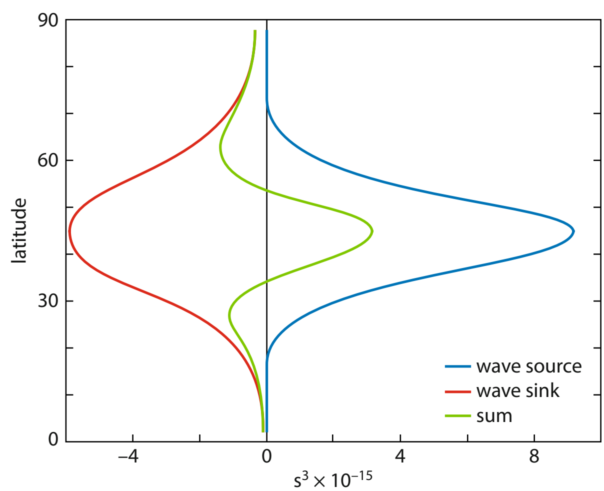

Fig. 9.28 As in Fig. 9.27, but now showing the wave source $\langle \zeta' F'_\zeta \rangle$, the wave sink $\langle \zeta' D'_\zeta \rangle$, and their net effect, agreeing with $r\langle u \rangle$. Redrawn from Vallis (2006) with permission from Cambridge University Press

9.3.3 A Two-Layer Model

The discussion above of purely barotropic dynamics does not admit the explicit treatment of a baroclinic wave source. Moreover, it also does not allow a description of an altitude-dependent mean circulation. The most simple framework to make this possible is a two-layer model.

The Model Equations

In the baroclinic perspective as well the mechanism of vorticity conservation pertains as it is described by the Kelvin theorem. Now however, as in Chap. 4.5.2, it is to be applied to isentropic material surface elements, and vorticity is to be replaced by potential vorticity. Therefore the most direct route from the barotropic to the baroclinic perspective is a model with two isentropic layers (Held, 2000). A bit easier, however, is the approach of Vallis (2006), where a model with two layers is considered that each have constant density. The resulting equations are the same as in the isentropic two-layer model. The conditions are sketched in Fig. 9.29: One has two layers with constant densities

$$\rho_1 = \rho_0 - \tilde{\rho}_1 \qquad \tilde{\rho}_1 \ll \rho_0 \tag{9.242}$$

$$\rho_2 = \rho_0 + \tilde{\rho}_2 \qquad \tilde{\rho}_2 \ll \rho_0 \tag{9.243}$$

that do not differ much. The upper lid is rigid, which is only possible if a temporally and spatially dependent pressure $p_T(x, y, t)$ is applied there. The interface between the two layers is variable so that its vertical displacement relative to the equilibrium position is $\eta(x, y, t)$. The respective layer thicknesses are $h_1(x, y, t)$ and $h_2(x, y, t)$, with corresponding equilibrium values H_1 and H_2. As we are interested in synoptic-scale eddies, these are in geostrophic and hydrostatic equilibrium so that η is small in comparison with H_1 and H_2. The total thickness of the model atmosphere is

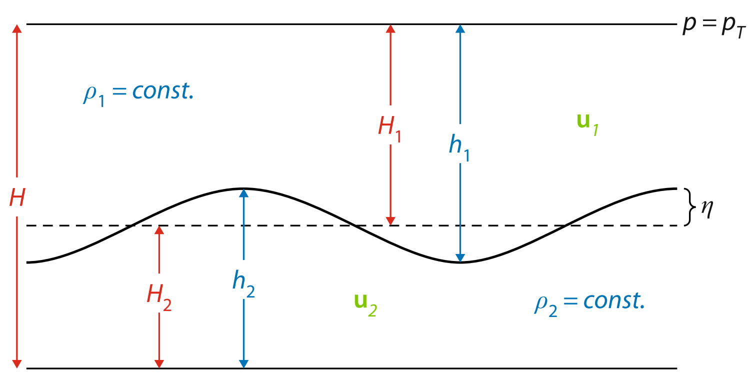

Fig. 9.29 Geometry of a two-layer model for the discussion of the mean circulation in midlatitudes

$$H = h_1 + h_2 = H_1 + H_2 = \text{const.} \tag{9.244}$$

As in the derivation of the shallow-water equations one assumes that the horizontal winds $\mathbf{u}_1$ and $\mathbf{u}_2$ in the layers do not depend on altitude.

The corresponding pressure results from hydrostatic equilibrium

$$\frac{\partial p_i}{\partial z} = -g\rho_i \tag{9.245}$$

i.e.,

$$p_1 = p_T + \rho_1 g(H - z) \tag{9.246}$$

$$p_2 = p_T + \rho_1 g(H_1 - \eta) + \rho_2 g(H_2 + \eta - z) \tag{9.247}$$

Hence the pressure-gradient accelerations acting in the two horizontal-momentum equations are

$$\frac{1}{\rho_1}\nabla_h p_1 = \frac{1}{\rho_1}\nabla p_T \approx \frac{1}{\rho_0}\nabla p_T \tag{9.248}$$

$$\frac{1}{\rho_2}\nabla_h p_2 = \frac{1}{\rho_2}\nabla p_T + g\frac{\rho_2 - \rho_1}{\rho_2}\nabla\eta \approx \frac{1}{\rho_0}\nabla p_T + g'\nabla\eta \tag{9.249}$$

Here we have assumed $\rho_i \approx \rho_0$ in the denominator. Moreover,

$$g' = g\frac{\rho_2 - \rho_1}{\rho_0} \tag{9.250}$$

is the so-called reduced gravity. Including simple turbulent friction in the lower layer, the horizontal-momentum equations becomes

$$\frac{D\mathbf{u}_1}{Dt} + f\mathbf{e}_z \times \mathbf{u}_1 = -\nabla\frac{p_T}{\rho_0} \tag{9.251}$$

$$\frac{D\mathbf{u}_2}{Dt} + f\mathbf{e}_z \times \mathbf{u}_2 = -\nabla\frac{p_T}{\rho_0} - g'\nabla\eta - r\mathbf{u}_2 \tag{9.252}$$

Here r is the turbulent drag coefficient.

The continuity equation in each layer is

$$\nabla \cdot \mathbf{u}_i + \frac{\partial w_i}{\partial z} = 0 \tag{9.253}$$

Elimination of the vertical wind is achieved as in shallow-water theory: Vertical integration, e.g., of the upper layer, first yields

$$h_1\nabla \cdot \mathbf{u}_1 + [w_1]_{H_1-\eta}^{H} = 0 \tag{9.254}$$

However

$$[w_1]_{H_1-\eta}^{H} = \left[\frac{Dz}{Dt}\right]_{H_1-\eta}^{H} = \frac{Dh_1}{Dt} \tag{9.255}$$

so that one obtains

$$\frac{Dh_1}{Dt} + h_1 \nabla \cdot \mathbf{u}_1 = 0 \tag{9.256}$$

or correspondingly

$$\frac{\partial h_1}{\partial t} + \nabla \cdot (h_1 \mathbf{u}_1) = 0 \tag{9.257}$$

The treatment of the lower layer is completely analogous, so that generally

$$\frac{\partial h_i}{\partial t} + \nabla \cdot (h_i \mathbf{u}_i) = 0 \qquad i = 1, 2 \tag{9.258}$$

For reasons that will become clearer farther below these equations shall also be extended by sources and sinks S_i describing a mass exchange between the two layers that is due to sinking and rising air masses. Hence

$$\boxed{\frac{\partial h_i}{\partial t} + \nabla \cdot (h_i \mathbf{u}_i) = S_i} \tag{9.259}$$

are used.

Geostrophic and Thermal Wind

The geostrophic equilibria between Coriolis force and pressure-gradient force are respectively

$$f_0 \mathbf{e}_z \times \mathbf{u}_1 = -\nabla \frac{p_T}{\rho_0} \tag{9.260}$$

$$f_0 \mathbf{e}_z \times \mathbf{u}_2 = -\nabla \left(\frac{p_T}{\rho_0} + g'\eta\right) \tag{9.261}$$

where f_0 is the value of the Coriolis parameter at a mean reference latitude, so that the geostrophic winds are

$$\mathbf{u}_{1g} = \mathbf{e}_z \times \frac{1}{f_0} \nabla \frac{p_T}{\rho_0} \tag{9.262}$$

$$\mathbf{u}_{2g} = \mathbf{e}_z \times \frac{1}{f_0} \nabla \left(\frac{p_T}{\rho_0} + g'\eta\right) \tag{9.263}$$

The difference between these yields the thermal-wind relation

$$\boxed{f_0 \left(\mathbf{u}_{1g} - \mathbf{u}_{2g}\right) = -g' \mathbf{e}_z \times \nabla \eta} \tag{9.264}$$

which is component-wise

$$f_0\left(u_{1g} - u_{2g}\right) = g'\frac{\partial \eta}{\partial y} \tag{9.265}$$

$$f_0\left(v_{1g} - v_{2g}\right) = -g'\frac{\partial \eta}{\partial x} \tag{9.266}$$

Hence, $-\eta$ takes the role of potential temperature. An interface slope as in Fig. 9.30, i.e., with increasing altitude from the equator to the pole, corresponds to a vertically increasing zonal wind. In order to be close to real atmospheric conditions one therefore needs source terms S_i transferring at the equator mass from the lower layer to the upper layer, as would result from direct heating, and transferring at the pole mass from the upper layer to the lower layer, corresponding to cooling there.

The Zonal-Mean Equations in Quasigeostrophic Scaling

In midlatitudes we can assume geostrophic winds

$$\mathbf{u}_i \approx \mathbf{u}_{ig} \tag{9.267}$$

Since the latter are non-divergent

$$\nabla \cdot \mathbf{u}_{ig} = 0 \tag{9.268}$$

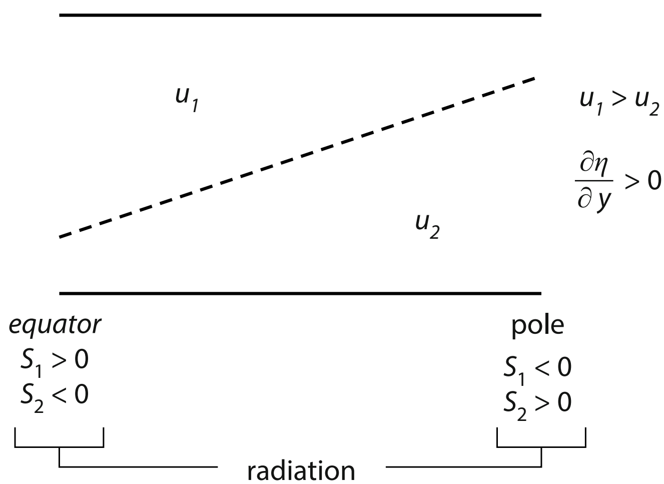

Fig. 9.30 The interface of the two-layer model is tilted upwards from the tropics to the pole, corresponding to a decrease of potential temperature and a positive zonal thermal wind. This tilt is due to source terms in the continuity equations that describe the effect of solar heating. The latter leads to rising air masses in the tropics and sinking air masses in the polar regions

the zonal-momentum equations can be written at good accuracy

$$\frac{\partial u_1}{\partial t} + \nabla \cdot (\mathbf{u}_1 u_1) - f v_1 = -\frac{\partial}{\partial x}\frac{p_T}{\rho_0} \tag{9.269}$$

$$\frac{\partial u_2}{\partial t} + \nabla \cdot (\mathbf{u}_2 u_2) - f v_2 = -\frac{\partial}{\partial x}\left(\frac{p_T}{\rho_0} + g'\eta\right) - r u_2 \tag{9.270}$$

The zonal mean of these is

$$\frac{\partial \langle u_1 \rangle}{\partial t} + \frac{\partial}{\partial y}\left\langle v_1' u_1' \right\rangle - f \langle v_1 \rangle = 0 \tag{9.271}$$

$$\frac{\partial \langle u_2 \rangle}{\partial t} + \frac{\partial}{\partial y}\left\langle v_2' u_2' \right\rangle - f \langle v_2 \rangle = -r \langle u_2 \rangle \tag{9.272}$$

As in barotropic theory one has also, however,

$$\frac{\partial}{\partial y}\left\langle v_i' u_i' \right\rangle \approx \frac{\partial}{\partial y}\left\langle v_{gi}' u_{gi}' \right\rangle = -\left\langle \zeta_{gi}' v_{gi}' \right\rangle \approx -\left\langle \zeta_i' v_i' \right\rangle \tag{9.273}$$

Moreover

$$f \langle v_i \rangle = f_0 \langle v_i \rangle + \mathcal{O}(Ro f_0 U) \tag{9.274}$$

because the β-term is only $\mathcal{O}(Ro)$ in comparison with the leading term. Hence the zonal-mean of the zonal-momentum equation becomes approximately

$$\frac{\partial \langle u_1 \rangle}{\partial t} - f_0 \langle v_1 \rangle = \left\langle v_1' \zeta_1' \right\rangle \tag{9.275}$$

$$\frac{\partial \langle u_2 \rangle}{\partial t} - f_0 \langle v_2 \rangle = \left\langle v_2' \zeta_2' \right\rangle - r \langle u_2 \rangle \tag{9.276}$$

Averaging the continuity equations likewise results in

$$\frac{\partial \langle h_i \rangle}{\partial t} + \frac{\partial}{\partial y}\left(\langle h_i \rangle \langle v_i \rangle\right) = -\frac{\partial}{\partial y}\langle h_i' v_i' \rangle + \langle S_i \rangle, \qquad i = 1, 2 \tag{9.277}$$

so that the *Eulerian mean* of the two-layer model is

$$\frac{\partial \langle u_1 \rangle}{\partial t} - f_0 \langle v_1 \rangle = \left\langle v_1' \zeta_1' \right\rangle \tag{9.278}$$

$$\frac{\partial \langle u_2 \rangle}{\partial t} - f_0 \langle v_2 \rangle = \left\langle v_2' \zeta_2' \right\rangle - r \langle u_2 \rangle \tag{9.279}$$

$$\frac{\partial \langle h_i \rangle}{\partial t} + \frac{\partial}{\partial y}\left(\langle h_i \rangle \langle v_i \rangle\right) = -\frac{\partial}{\partial y}\langle h_i' v_i' \rangle + \langle S_i \rangle, \qquad i = 1, 2 \tag{9.280}$$

Herein $\langle h_i \rangle \approx H_i$ because η is small, an approximation not applicable in meridional derivatives of the zonal-mean thicknesses.

Now for the transformed Eulerian mean. First, as in shallow-water theory one can easily show that in the absence of friction ($r = 0$) and sinks and sources ($S_1 = S_2 = 0$)

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_i \cdot \nabla\right) \Pi_i = 0 \tag{9.281}$$

where

$$\Pi_i = \frac{\zeta_i + f}{h_i} \tag{9.282}$$

is the ith-layer potential vorticity of the two-layer model. In synoptic scaling one has

$$|\eta| \ll h_i \tag{9.283}$$

$$|\zeta_i| \ll f_0 \tag{9.284}$$

so that $\Pi_i \approx \pi_i / H_i$, where

$$\pi_i = \zeta_i + f - f_0 \frac{h_i - H_i}{H_i} \tag{9.285}$$

is the corresponding quasigeostrophic potential vorticity. Hence

$$\left\langle v_i' \pi_i' \right\rangle = \left\langle v_i' \zeta_i' \right\rangle - f_0 \frac{\left\langle v_i' h_i' \right\rangle}{H_i} \tag{9.286}$$

However, the mass of a fluid column is proportional to its thickness so that a mass-weighted (transformed Eulerian) mean of the meridional velocity in each layer is

$$\langle v_i \rangle_* = \frac{\langle v_i h_i \rangle}{\langle h_i \rangle} = \langle v_i \rangle + \frac{\left\langle v_i' h_i' \right\rangle}{\langle h_i \rangle} \tag{9.287}$$

because

$$\langle h_i v_i \rangle = \langle h_i \rangle \langle v_i \rangle + \langle h_i' v_i' \rangle \tag{9.288}$$

Hence, and because $\langle h_i \rangle \approx H_i$, the zonal-mean momentum equations become in good approximation

$$\frac{\partial \langle u_1 \rangle}{\partial t} - f_0 \langle v_1 \rangle_* = \langle v_1' \pi_1' \rangle \tag{9.289}$$

$$\frac{\partial \langle u_2 \rangle}{\partial t} - f_0 \langle v_2 \rangle_* = \langle v_2' \pi_2' \rangle - r \langle u_2 \rangle \tag{9.290}$$

Likewise the zonal-mean continuity equations can be re written

$$\frac{\partial \langle h_i \rangle}{\partial t} + \frac{\partial}{\partial y} \left(\langle h_i \rangle \langle v_i \rangle_* \right) = \langle S_i \rangle, \qquad i = 1, 2 \tag{9.291}$$

so that in summary the *transformed Eulerian mean (TEM)* of the two-layer model is

$$\frac{\partial\langle u_1\rangle}{\partial t} - f_0\langle v_1\rangle_* = \langle v_1'\pi_1'\rangle \tag{9.292}$$

$$\frac{\partial\langle u_2\rangle}{\partial t} - f_0\langle v_2\rangle_* = \langle v_2'\pi_2'\rangle - r\langle u_2\rangle \tag{9.293}$$

$$\frac{\partial\langle h_i\rangle}{\partial t} + \frac{\partial}{\partial y}\left(\langle h_i\rangle\langle v_i\rangle_*\right) = \langle S_i\rangle, \qquad i = 1, 2 \tag{9.294}$$

Integral Properties

From the equations follow two important properties of vertical integrals of the model. First, due to the non-divergence of the approximately geostrophic wind, the continuity equations can also be written

$$\frac{\partial h_i}{\partial t} + \mathbf{u}_i \cdot \nabla h_i = S_i, \qquad i = 1, 2 \tag{9.295}$$

Inserting

$$h_1 = H_1 - \eta \tag{9.296}$$

$$h_2 = H_2 + \eta \tag{9.297}$$

yields

$$-\frac{\partial\eta}{\partial t} - \mathbf{u}_1 \cdot \nabla\eta = S_1 \tag{9.298}$$

$$\frac{\partial\eta}{\partial t} + \mathbf{u}_2 \cdot \nabla\eta = S_2 \tag{9.299}$$

The sum of these two equations is

$$-(\mathbf{u}_1 - \mathbf{u}_2)\cdot\nabla\eta = S_1 + S_2 \tag{9.300}$$

Due to the thermal-wind relationship (9.264), however, the left-hand side of this equation vanishes so that

$$S_1 + S_2 = 0 \tag{9.301}$$

and there is a mass-exchange term S from which S_1 and S_2 can be determined via

$$S_2 = S \tag{9.302}$$

$$S_1 = -S \tag{9.303}$$

where

$$S(y = 0) < 0 \qquad S(y = L_y) > 0 \qquad \frac{dS}{dy} \geq 0 \tag{9.304}$$

Inserting this into the transformed Eulerian mean of the continuity equations yields

$$-\frac{\partial\langle\eta\rangle}{\partial t}+\frac{\partial}{\partial y}\left(\langle h_1\rangle\langle v_1\rangle_*\right)=-\langle S\rangle \tag{9.305}$$

$$\frac{\partial\langle\eta\rangle}{\partial t}+\frac{\partial}{\partial y}\left(\langle h_2\rangle\langle v_2\rangle_*\right)=\langle S\rangle \tag{9.306}$$

so that in sum

$$\frac{\partial}{\partial y}\left(\langle h_1\rangle\langle v_1\rangle_*+\langle h_2\rangle\langle v_2\rangle_*\right)=0 \tag{9.307}$$

Once more we apply the meridional boundary conditions

$$y=0, L_y: \qquad v_i=0 \tag{9.308}$$

so that

$$\langle h_1\rangle\langle v_1\rangle_*+\langle h_2\rangle\langle v_2\rangle_*=0 \tag{9.309}$$

or, due to $h_i \approx H_i$,

$$\boxed{H_1\langle v_1\rangle_*+H_2\langle v_2\rangle_*=0} \tag{9.310}$$

Hence the vertical integral of the mass flux vanishes.

Moreover, from the relationship (9.266) for the meridional component of the thermal wind follows at good accuracy

$$\eta'\left(v_1'-v_2'\right)=-\frac{g'}{f_0}\frac{\partial}{\partial x}\frac{\eta'^2}{2} \tag{9.311}$$

so that in the zonal mean

$$\langle\eta'\left(v_1'-v_2'\right)\rangle=0 \tag{9.312}$$

Because due to (9.296) and (9.297)

$$\eta'=-h_1'=h_2' \tag{9.313}$$

one obtains from this

$$\langle v_1'h_1'\rangle+\langle v_2'h_2'\rangle=0 \tag{9.314}$$

i.e., in the vertical mean the eddies do not transport any mass. Hence the approximate vertical integral of the potential-vorticity fluxes is

$$\begin{aligned}
H_1\langle v_1'\pi_1'\rangle+H_2\langle v_2'\pi_2'\rangle &= H_1\left\langle v_1'\left(\zeta_1'-\frac{f_0}{H_1}h_1'\right)\right\rangle+H_2\left\langle v_2'\left(\zeta_2'-\frac{f_0}{H_2}h_2'\right)\right\rangle\\
&= H_1\langle v_1'\zeta_1'\rangle+H_2\langle v_2'\zeta_2'\rangle\\
&= -\frac{\partial}{\partial y}\left(H_1\langle u_1'v_1'\rangle+H_2\langle u_2'v_2'\rangle\right)
\end{aligned} \tag{9.315}$$

where we have used (9.273) in the last step. Because, due to the meridional boundary conditions of the β-channel, there are no momentum fluxes, one finally obtains

$$\boxed{\int_0^{L_y} dy\,\left(H_1\langle v_1'\pi_1'\rangle + H_2\langle v_2'\pi_2'\rangle\right) = 0} \tag{9.316}$$

Hence the vertical and meridional mean of the potential-vorticity flux vanishes.

The Dynamics of the Climatological Mean

Based on the relationships derived above, we now discuss the dynamics of a climatological mean, where all time derivatives disappear by time averaging and where all zonal means are replaced by zonal and temporal means. First, integration of the climatological means of (9.305) and (9.306) yield, again using the meridional boundary conditions for v_i,

$$\langle v_1\rangle_* \approx -\frac{1}{H_1}\int_0^y \langle S\rangle \tag{9.317}$$

$$\langle v_2\rangle_* \approx \frac{1}{H_2}\int_0^y \langle S\rangle \tag{9.318}$$

Due to (9.304), however, this means that

$$\boxed{\langle v_1\rangle_* > 0} \tag{9.319}$$

$$\boxed{\langle v_2\rangle_* < 0} \tag{9.320}$$

i.e., the residual circulation in the upper layer is directed to the pole and the one in the lower layer to the equator. Effectively this is a consequence of mass conservation together with heating (cooling) in the tropics (polar regions).

Next, the approximate vertical integral of the zonal-momentum equations in the TEM yields

$$-f_0\left(H_1\langle v_1\rangle_* + H_2\langle v_2\rangle_*\right) = H_1\langle v_1'\pi_1'\rangle + H_2\langle v_2'\pi_2'\rangle - r H_2\langle u_2\rangle \tag{9.321}$$

or with (9.310)

$$\boxed{r H_2\langle u_2\rangle = H_1\langle v_1'\pi_1'\rangle + H_2\langle v_2'\pi_2'\rangle} \tag{9.322}$$

i.e., the mean zonal wind in the lower layer (the surface wind of this model) agrees with the vertical integral of the potential-vorticity flux. The resulting wind distribution hence only depends on the corresponding balance. Now consider the climatological mean of the upper-layer zonal-momentum equation in the TEM. One obtains

$$-f_0\langle v_1\rangle_* = \langle v_1'\pi_1'\rangle \tag{9.323}$$

Hence

$$\langle v_1' \pi_1' \rangle < 0 \tag{9.324}$$

and therefore, due to (9.316), the potential-vorticity flux in the lower layer tends to be positive. For a clearer picture of the balance between the two fluxes recall that the meridional group velocity of the potential-vorticity-transporting waves is

$$c_{gy} \propto \frac{\partial \langle \pi_i \rangle}{\partial y} = -\frac{\partial^2 \langle u_i \rangle}{\partial y^2} + \beta - \frac{f_0}{H_i} \frac{\partial \langle h_i \rangle}{\partial y} \tag{9.325}$$

Typically, however, the relative-vorticity gradient is smaller than the planetary-vorticity gradient,

$$\left| \frac{\partial^2 \langle u_i \rangle}{\partial y^2} \right| < \beta \tag{9.326}$$

so that roughly

$$\frac{\partial \langle \pi_1 \rangle}{\partial y} \approx \beta - \frac{f_0}{H_1} \frac{\partial \langle h_1 \rangle}{\partial y} \tag{9.327}$$

$$\frac{\partial \langle \pi_2 \rangle}{\partial y} \approx \beta - \frac{f_0}{H_2} \frac{\partial \langle h_2 \rangle}{\partial y} \tag{9.328}$$

However, due to solar heating

$$\frac{\partial \langle h_1 \rangle}{\partial y} < 0 \tag{9.329}$$

$$\frac{\partial \langle h_2 \rangle}{\partial y} > 0 \tag{9.330}$$

so that

$$\frac{\partial \langle \pi_1 \rangle}{\partial y} > \beta \gg 0 \tag{9.331}$$

$$\frac{\partial \langle \pi_2 \rangle}{\partial y} \lesssim 0 \tag{9.332}$$

Here the result for the lower layer cannot be understood without the comment that, from a calculation as in the derivation of the Rayleigh theorem in Sect. 6.4.2, left to the interested reader as an exercise, baroclinic instability is only possible if

$$\sum_{i=1}^{2} H_i \int_0^{L_y} dy \frac{|\psi_i|^2}{|\omega - k\langle u_i \rangle|^2} \frac{\partial \langle \pi_i \rangle}{\partial y} = 0 \tag{9.333}$$

Here ω is the complex eigenfrequency of the baroclinic instability, $\hat{\psi}_i$ the corresponding streamfunction amplitude in the ith layer, where the quasigeostrophic streamfunctions are approximately

$$\psi_i = \frac{1}{f_0} \frac{p_T}{\rho_0} \tag{9.334}$$

$$\psi_2 = \frac{1}{f_0} \left(\frac{p_T}{\rho_0} + g'\eta \right) \tag{9.335}$$

so that

$$\eta = -\frac{f_0}{g'} (\psi_1 - \psi_2) \tag{9.336}$$

For the two-layer model to generate waves at all, (9.333) must be fulfilled, and hence the lower-layer potential-vorticity gradient must be negative at some locations. In this aspect the model differs perhaps from reality. The climatological gradient of potential vorticity from observation is shown in Fig. 9.31, and one sees that the gradient is predominantly positive at all altitudes. The essential result remains, however, so that the upper-layer Rossby waves are faster and hence are better able to spread potential-vorticity fluctuations. Therefore it is to be expected that the distribution of $\langle v_1' \pi_1' \rangle < 0$ is broader than that of $\langle v_2' \pi_2' \rangle > 0$. Because of (9.316), however, the areas under the two distributions agree with each other, so that

$$\begin{array}{l} \langle u_2 \rangle > 0 \text{ in middle latitudes} \\ \langle u_2 \rangle < 0 \text{ at the flanks of the jet stream} \end{array} \tag{9.337}$$

This is sketched in Fig. 9.32 and agrees with the empirical findings. The upper-layer winds result therefrom and the baroclinic shear of the zonal-mean atmosphere according to the thermal-wind equation (9.264). *It is important to realize that it is the surface winds that are controlled by the eddies together with surface friction, and that the upper-tropopsheric winds simply result from those!*

Finally, the upper branch of the Ferrel circulation is obtained by considering the climatological and Eulerian-mean zonal-momentum equation

$$-\ f_0 \langle v_1 \rangle = \langle v_1' \zeta_1' \rangle \tag{9.338}$$

As in barotropic dynamics, e.g., via the relationship between Rossby-wave group velocity and momentum flux, we have here as well

$$\langle v_1' \zeta_1' \rangle > 0 \tag{9.339}$$

Hence

$$\langle v_1 \rangle < 0 \tag{9.340}$$

In the lower layer, however, the eddy fluctuations are weak so that $\langle v_2' \zeta_2' \rangle$ can be neglected in the corresponding Eulerian-mean zonal-momentum equation, whence the climatological mean of the latter is

$$-\ f_0 \langle v_2 \rangle \approx -r \langle u_2 \rangle \tag{9.341}$$

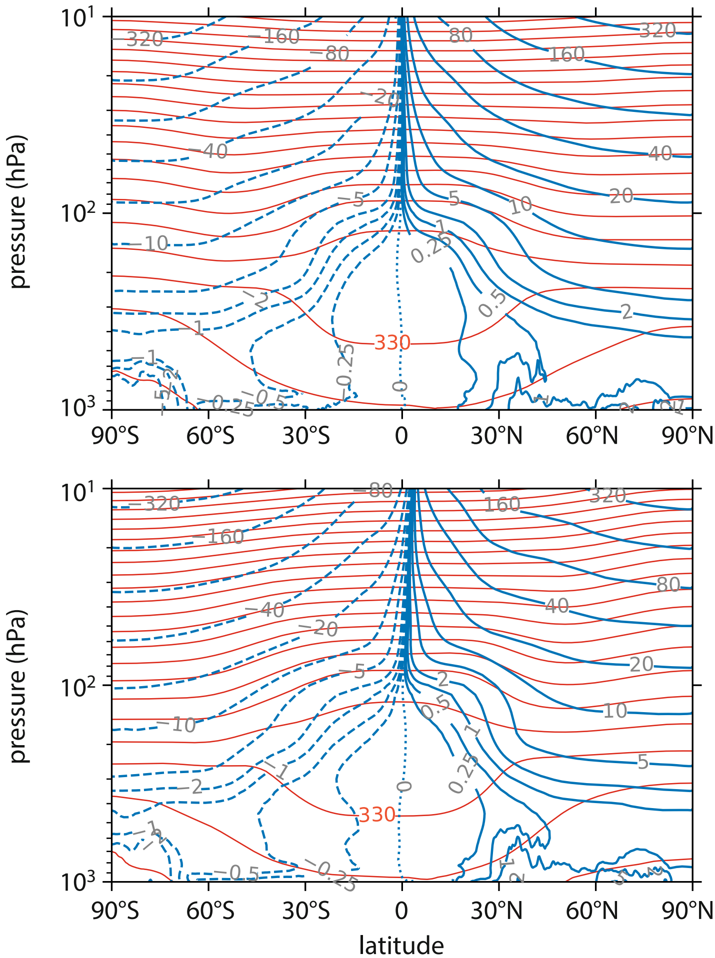

Fig. 9.31 Zonal-mean potential vorticity (blue contour lines) and zonal-mean potential temperature (red) in the annual mean (top panel) and in northern-hemispheric winter (bottom) from ERA5 analysis data (Hersbach et al., 2020)

requiring that in middle latitudes $\langle v_2 \rangle > 0$. In summary, one obtains the *Ferrel cell*

$$\text{in middle latitudes: } \begin{array}{l} \langle v_1 \rangle < 0 \\ \langle v_2 \rangle > 0 \end{array} \tag{9.342}$$

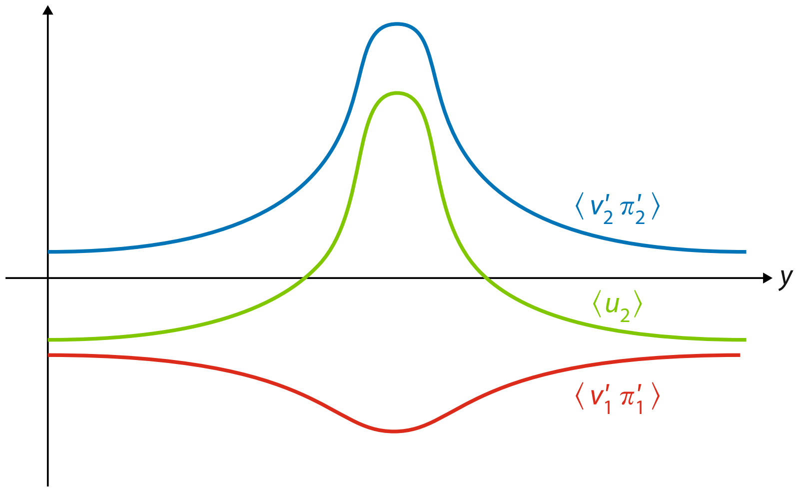

Fig. 9.32 Mid latitude surface winds are due to the balance between potential-vorticity fluxes in both layers, resulting in westerlies in middle latitudes, and easterlies at the corresponding flanks

9.3.4 The Continuously Stratified Atmosphere

Based on the discussions above, the representation of the mid latitude circulation in the continuously stratified atmosphere shall be done relatively quickly. A few parallels, but also differences, shall be described here in addition.

The Surface Winds

The connection between the surface winds and the potential-vorticity fluxes is similar to the one in the two-layer model. We recall that the TEM zonal-momentum equation is within the quasigeostrophic approximation

$$\frac{\partial \langle u \rangle}{\partial t} - f_0 \langle v \rangle^* = \langle v' \pi' \rangle + \langle F \rangle \tag{9.343}$$

where

$$\langle v \rangle^* = \langle v \rangle - \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \left(\frac{\overline{\rho}}{N^2} \langle v' b' \rangle \right) \tag{9.344}$$

is the residual meridional wind, and where turbulent friction is approximated via

$$\langle F \rangle = \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \left(\overline{\rho} K \frac{\partial \langle u \rangle}{\partial z} \right) \tag{9.345}$$

The climatological mean yields

$$-\overline{\rho} f_0 \langle v \rangle^* = \overline{\rho} \langle v' \pi' \rangle + \overline{\rho} \langle F \rangle \tag{9.346}$$

or, after inserting (9.344)

$$- f_0 \overline{\rho} \langle v \rangle + f_0 \frac{\partial}{\partial z} \left(\frac{\overline{\rho}}{N^2} \langle v' b' \rangle \right) = \overline{\rho} \langle v' \pi' \rangle + \frac{\partial}{\partial z} \left(\overline{\rho} K \frac{\partial \langle u \rangle}{\partial z} \right) \tag{9.347}$$

The vertical integral of this equation, defined for an arbitrary field X by

$$\{X\} = \int_0^\infty dz\, X \tag{9.348}$$

yields

$$- f_0 \{\overline{\rho} \langle v \rangle\} + f_0 \left[\frac{\overline{\rho}}{N^2} \langle v' b' \rangle \right]_0^\infty = \{\overline{\rho} \langle v' \pi' \rangle\} + \left[\overline{\rho} K \frac{\partial \langle u \rangle}{\partial z} \right]_0^\infty \tag{9.349}$$

In quasigeostrophic scaling the continuity equation is

$$\nabla \cdot (\overline{\rho} \mathbf{u}) + \frac{\partial}{\partial z} (\overline{\rho} w) = 0 \tag{9.350}$$

leading in the zonal mean to

$$\frac{\partial}{\partial y} (\overline{\rho} \langle v \rangle) + \frac{\partial}{\partial z} (\overline{\rho} \langle w \rangle) = 0 \tag{9.351}$$

Vertical integration gives

$$\frac{\partial}{\partial y} \{\overline{\rho} \langle v \rangle\} = 0 \tag{9.352}$$

due to the boundary conditions

$$\overline{\rho} \xrightarrow[z \to \infty]{} 0 \tag{9.353}$$

$$\langle w \rangle|_{z=0} = 0 \tag{9.354}$$

Since $\langle v \rangle = 0$ holds approximately at the poles ones has in general

$$\{\overline{\rho} \langle v \rangle\} = 0 \tag{9.355}$$

Moreover, likewise,

$$\left[\frac{\overline{\rho}}{N^2} \langle v' b' \rangle \right]_0^\infty = - \left[\frac{\overline{\rho}}{N^2} \langle v' b' \rangle \right]_{z=0} \tag{9.356}$$

For the negative turbulent momentum flux we assume

$$\overline{\rho} K \frac{\partial \langle u \rangle}{\partial z} = \begin{cases} 0 & z \to \infty \\ r \overline{\rho} \langle u \rangle|_{z=0} & z = 0 \end{cases} \tag{9.357}$$

so that one obtains in total

$$z = 0: \qquad r\langle u\rangle = \frac{f_0}{N^2}\langle v'b'\rangle + \frac{1}{\overline{\rho}}\left\{\overline{\rho}\langle v'\pi'\rangle\right\} \tag{9.358}$$

In the continuously stratified atmosphere as well, the surface winds result from the vertical integral of the potential-vorticity fluxes, however substituted by a contribution from the surface buoyancy fluxes.

The Potential-Vorticity Flux

Because the zonal-mean meridional potential-vorticity flux agrees with the divergence of the Eliassen–Palm flux,

$$\langle v'\pi'\rangle = \frac{1}{\overline{\rho}}\nabla\cdot\boldsymbol{\mathcal{F}} \tag{9.359}$$

$$\boldsymbol{\mathcal{F}} = -\overline{\rho}\langle u'v'\rangle\mathbf{e}_y + \overline{\rho}\frac{f_0}{N^2}\langle v'b'\rangle\mathbf{e}_z \tag{9.360}$$

consider the latter in Fig. 9.33. One sees a dominance of the vertical component in the mid latitude lower troposphere. This is the meridional heat flux resulting there from baroclinic instability. Because the Eliassen–Palm flux is also the flux of wave-action density, one sees that the latter is transported upwards and then equatorwards. The upwards increasing horizontal component is explained by the upwards increase of the meridional gradient in the zonal-mean potential vorticity (see Fig. 9.31) and hence also the meridional Rossby-wave group velocity. The signs of the Eliassen–Palm-flux divergence are such that it is negative in the upper troposphere and positive in the lower troposphere. This agrees well with the findings from the two-layer model, and it can also be identified in Fig. 9.34. One can see there as well, with an additional glance at Fig. 8.9, that the upper-troposphere wave fluxes

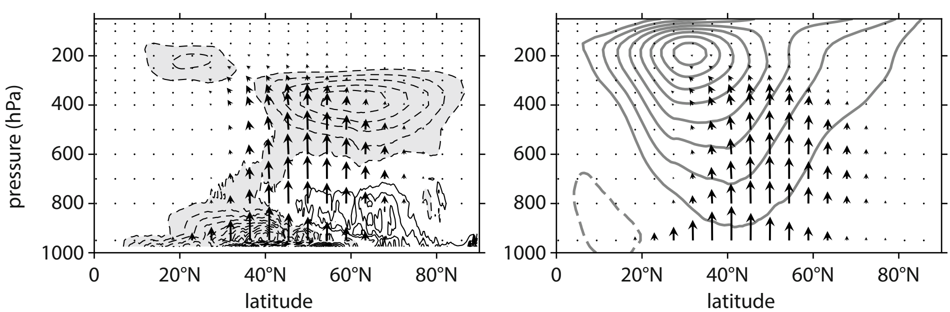

Fig. 9.33 From ERA5 analysis data (Hersbach et al., 2020), the Eliassen–Palm flux and its divergence (left panel, negative values indicated by dashed contours) together with the zonal-mean wind (right), both for the northern hemisphere in northern-hemispheric winter

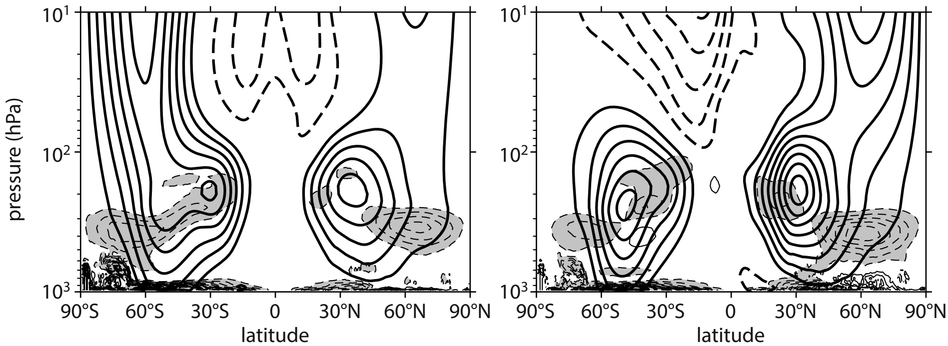

Fig. 9.34 From ERA5 analysis data (Hersbach et al., 2020), the Eliassen–Palm-flux divergence (thin contours, negative values indicated by shading) and the zonal-mean zonal wind (fat contours) in the yearly mean (left panel) and in northern-hemispheric winter (right)

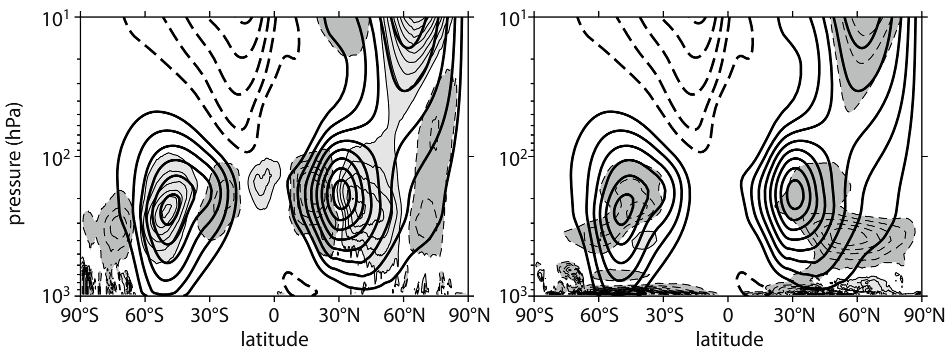

Fig. 9.35 From ERA5 analysis data (Hersbach et al., 2020) for northern-hemispheric winter: The decomposition of the Eliassen–Palm-flux divergence into its horizontal part (momentum-flux convergence, left panel) and its vertical part (heat flux, right), each together with the zonal-mean zonal wind as in Fig. 9.34

balance the effect of a poleward residual circulation, and that the lower-troposphere fluxes do so as well (in parts) for the equatorward residual circulation there. The decomposition of the Eliassen–Palm flux divergence into the respective contributions from the momentum and heat fluxes, shown in Fig. 9.35, exhibits quite clearly the pattern of momentum-flux convergence in midlatitudes and divergence at the flanks of the jet stream, as could already be predicted from barotropic theory. Note again that it is not the case at all that positive Eliassen–Palm-flux divergence leads to westerlies and negative Eliassen–Palm flux divergence to easterlies! This is because we are considering climatological means where all time derivatives are averaged out. A more intricate situation arises where the surface winds are controlled by a boundary-layer balance between turbulent friction and the vertical-mean Eliassen–Palm flux convergence, supplemented by surface buoyancy fluxes. The upper-tropospheric winds follow from the surface winds and thermal-wind balance.

9.3.5 Summary

Different to the tropics *planetary-and synoptic-scale waves* in middle latitudes, generated by the land–sea contrast of atmospheric heating, by orography, and by the process of baroclinic instability, are not just one additional factor in the explanation of the mean flow. Here they are *essential* from the beginning.

- *Phenomenologically* the meridional circulation in the Eulerian mean can be determined by the *solution of the elliptic equation for the mass streamfunction.* The vertical derivative of the momentum-flux convergence and the latitudinal derivative of the buoyancy-flux convergence are structured such that they force in midlatitudes indirect Ferrel cells. *At the surface one obtains westerlies*, needed for balancing in the Eulerian-mean meridional momentum equation, via friction, the poleward circulation at the ground. *Understanding* the wave-flux signs is, however, not possible without the *following arguments.*
- A *conspicuous mid latitude phenomenon is a jet stream with a considerably more barotropic structure* than the subtropical jet stream. This barotropic jet stream is forced by waves. An important factor is the *mixing of (potential) vorticity.* First clues to this can already be provided by a barotropic model. The conservation of absolute vorticity due to the *Kelvin theorem* explains that a mid latitude wave source unavoidably leads to momentum-flux convergence there, and hence a *westerly jet stream flanked by easterlies.* Alternatively this can be explained by the *connection between of the meridional group velocity and the momentum flux of Rossby waves* radiated by the wave source.
- The most simple framework for the study of *non-barotropic aspects,* i.e., the wave source, latitude-dependent temperature, baroclinicity of the winds, and the altitude-dependent meridional circulation is a *two-layer model,* where the *role of vorticity is taken by potential vorticity.* Essential are *sinks and sources* in the two continuity equations in this model capturing the *rising (sinking) of air masses due to heating in the tropics (cooling at the poles).* The residual circulation in the TEM of this model is the mass-weighted meridional flow. Due to mass conservation there is no vertical-mean flow. *In the upper troposphere air masses move polewards while they move equatorwards in the lower troposphere*. Due to the thermal wind potential-vorticity fluxes are balanced in the vertical and meridional mean. The *upper-layer potential-vorticity flux must be directed polewards* in order to balance in the climatological-mean zonal-momentum equation the Coriolis acceleration due to the meridional circulation. Hence the *lower-layer potential-vorticity flux must be directed equatorwards.* In the climatological mean of the zonal-momentum equation the

vertical integral of the potential-vorticity fluxes must be balanced by surface friction. This determines structure and sign of the surface winds. The different mean potential-vorticity gradients lead to *differences in the group velocities of the Rossby waves so that the positive upper-layer potential-vorticity flux has a broader distribution* than the oppositely directed flux in the lower layer. Hence one obtains in midlatitudes *surface westerlies flanked by easterlies.* It is the surface winds that are forced directly by the waves, while the upper-layer jet stream follows from these by thermal-wind balance. From the above the *Ferrel circulation* can be understood as well, via balancing of the Eulerian-mean momentum equation, without needing diagnostic input from analyses.
- Simulations and analyses of the *continuously stratified atmosphere* essentially support the role of the mechanisms discussed above. Again the surface winds follow from a balance between the vertical-mean potential-vorticity fluxes and surface friction. Here, however, *surface buoyancy fluxes* contribute as well.

9.4 Recommendations for Further Reading

The textbooks by Andrews et al. (1987), Holton and Hakim (2013), Lindzen (1990), and Vallis (2006) can all be helpful in deepening the material in this chapter. The same holds for the original publications by Schneider (1977), Held and Hou (1980) and Lindzen and Hou (1988). Sources on the effect of Rossby waves on the Hadley circulation are Becker et al. (1997), Vallis (2006), and Walker and Schneider (2006). The discussion of the circulation in midlatitudes is based on Held (2000) and Vallis (2006). Useful sources on the energetics of the general circulation are Lorenz (1967) and the textbooks of Peixoto and Oort (1992) and Hartmann (2016).