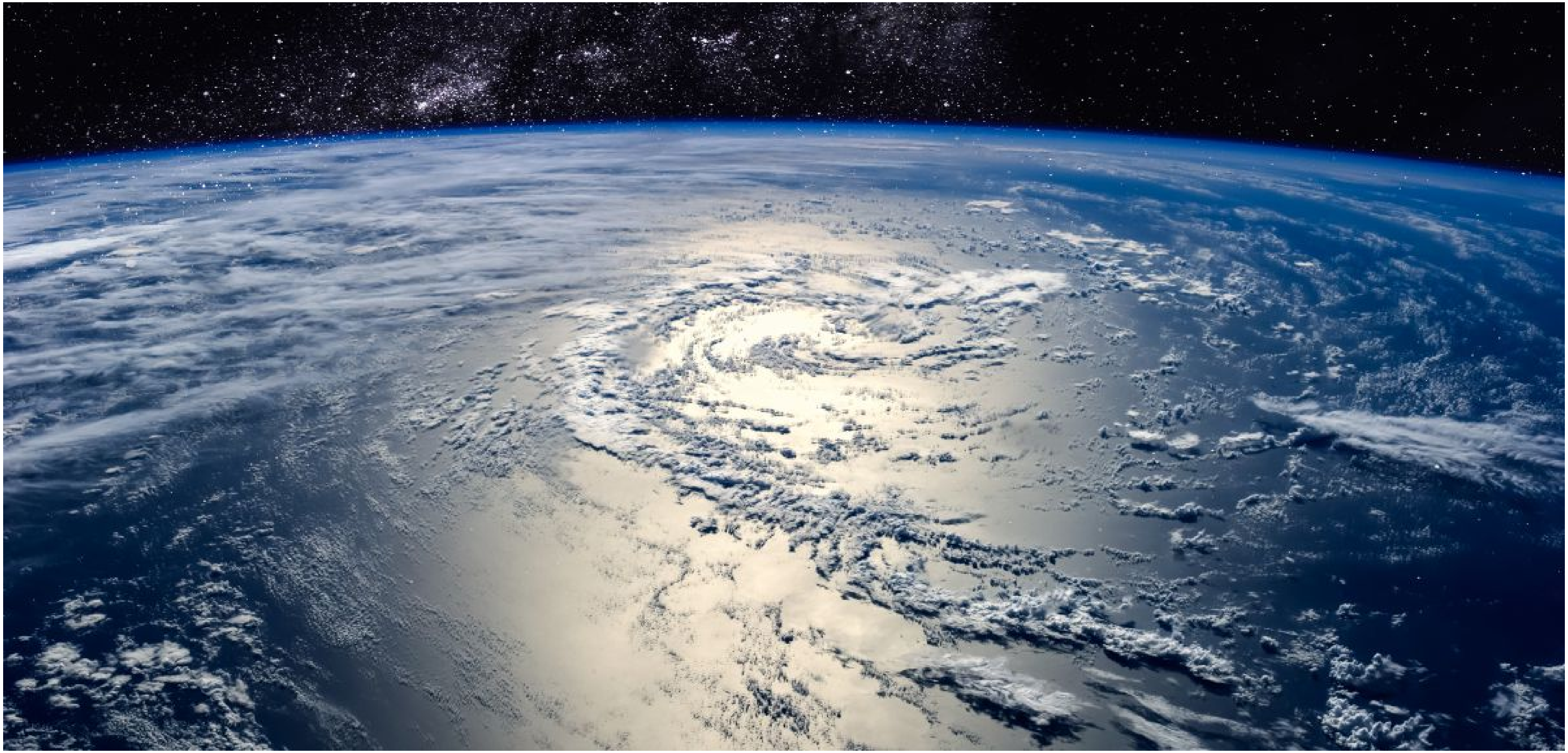
A multilayer model for non-hydrostatic multiscale free-surface flows

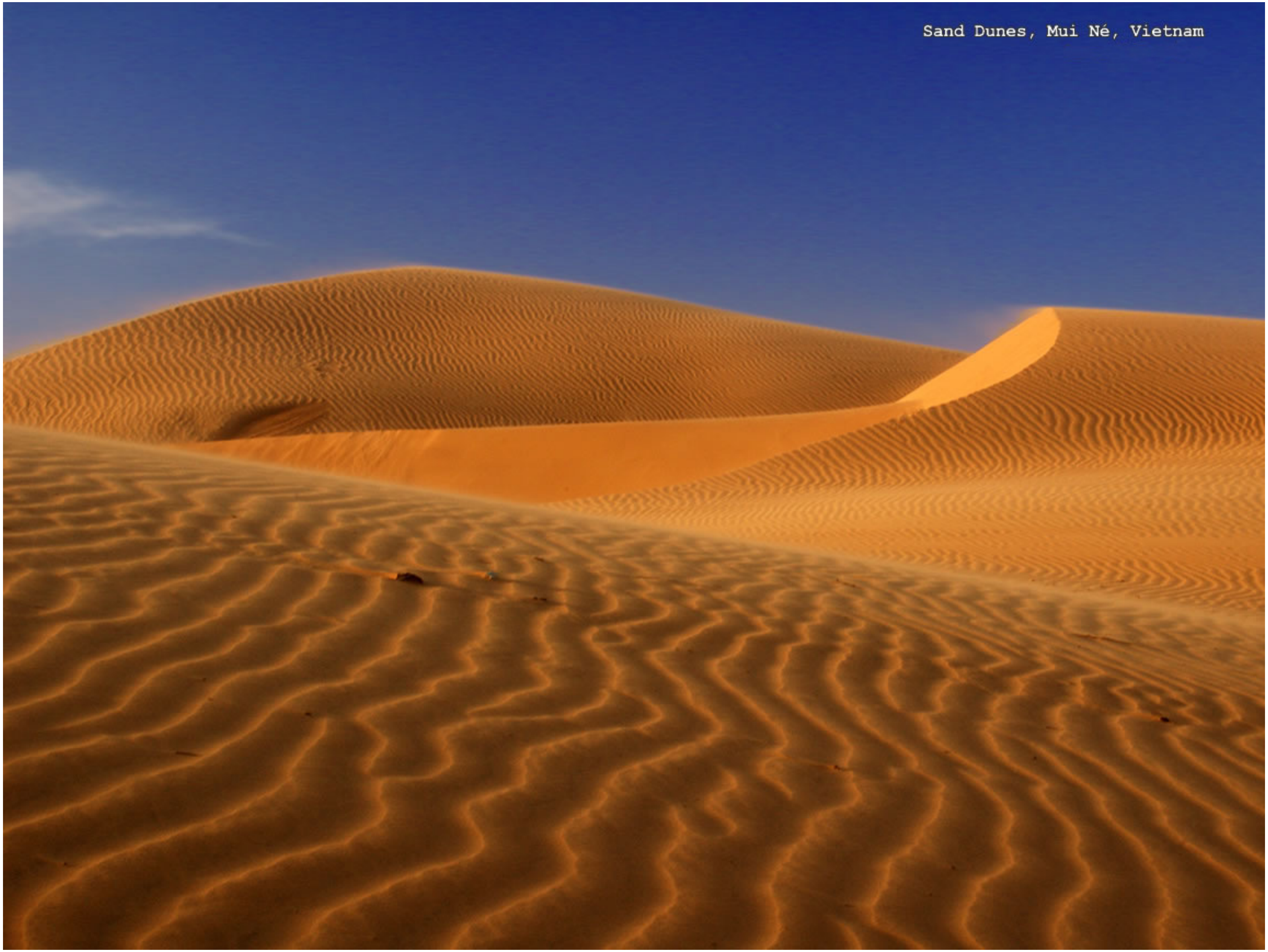
Stéphane Popinet

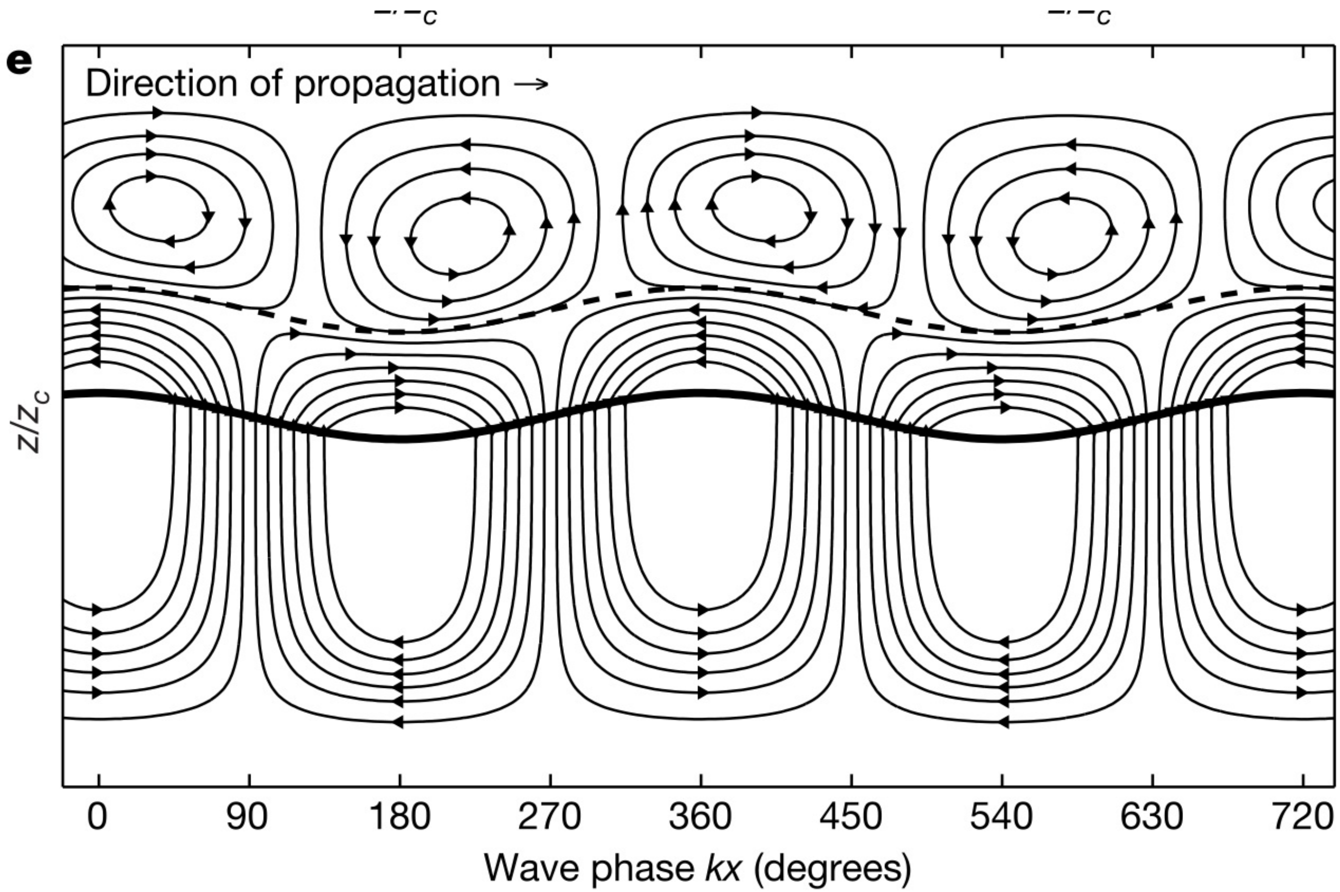
Institut ∂'Alembert

CNRS / Sorbonne Université

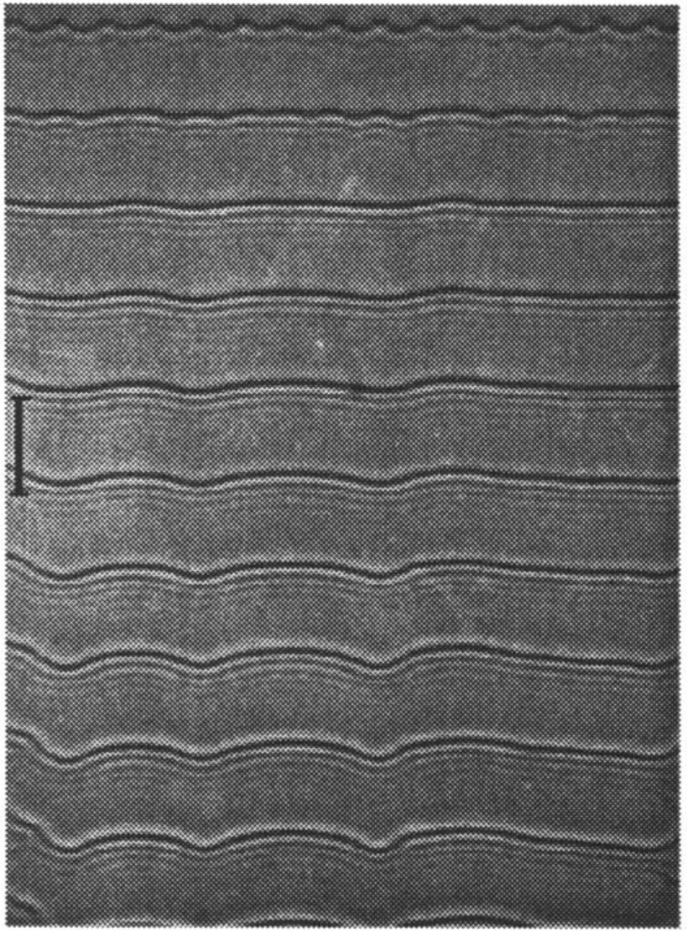
∂'Alembert
Institut Jean le Rond d'Alembert

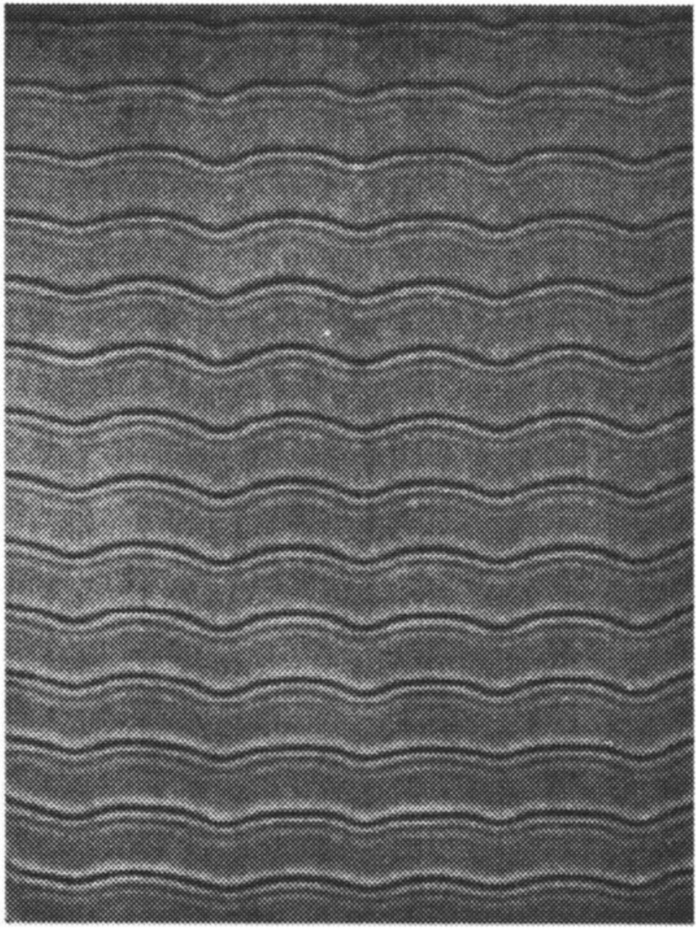



Sand Dunes, Mui Né, Vietnam



Hristov et al, *Dynamical coupling of wind and ocean waves through wave-induced air flow*, Nature, 2003.





Piotr Leonidovich Kapitza, "Wave flow of thin layers of a viscous fluid," in Collected papers of P.L. Kapitza, D. Ter Haar, Ed., pp. 662–689. Pergamon, 1948.

$$
\begin{aligned}
\partial_t h + \boldsymbol{\nabla} \cdot (h\,\mathbf{u}) &= 0, \\
\partial_t (h\,\mathbf{u}) + \boldsymbol{\nabla} \cdot (h\,\mathbf{u}\,\mathbf{u}) &= -g\,h\,\boldsymbol{\nabla}\eta, \\
\eta &= z_b + h
\end{aligned}
$$

Adhémar Jean-Claude Barré de Saint-Venant, 1871.

- \+ Simple
- \+ System of conservation laws
- \+ Fully non-linear
- \+ Numerically efficient
- − Vertical structure is entirely modelled
- − Hydrostatic, no-vertical acceleration/momentum
- Non-dispersive waves: $c = \sqrt{g\,h}$

1797–1886

$$
\begin{aligned}
\partial_t h + \boldsymbol{\nabla} \cdot (h\,\mathbf{u}) &= 0, \\
\partial_t (h\,\mathbf{u}) + \boldsymbol{\nabla} \cdot (h\,\mathbf{u}\,\mathbf{u}) &= -g\,h\,\boldsymbol{\nabla}\eta, \\
\eta &= z_b + h
\end{aligned}
$$

Adhémar Jean-Claude Barré de Saint-Venant, 1871.

+ Simple
+ System of conservation laws
+ Fully non-linear
+ Numerically efficient
− Vertical structure is entirely modelled
− Hydrostatic, no-vertical acceleration/momentum

Non-dispersive waves: $c = \sqrt{g\,h}$

1797–1886

$$
\begin{aligned}
\partial_t h + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}) &= 0,\\
\partial_t(h\,\mathbf{u}) + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}\,\mathbf{u}) &= -g\,h\,\boldsymbol{\nabla}\eta,\\
\eta &= z_b + h
\end{aligned}
$$

Adhémar Jean-Claude Barré de Saint-Venant, 1871.

+ Simple
+ System of conservation laws
+ Fully non-linear
+ Numerically efficient
− Vertical structure is entirely modelled
− Hydrostatic, no-vertical acceleration/momentum

Non-dispersive waves: $c = \sqrt{g\,h}$

1797–1886

$$\begin{aligned}\partial_t h + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}) &= 0,\\ \partial_t(h\,\mathbf{u}) + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}\,\mathbf{u}) &= -g\,h\,\boldsymbol{\nabla}\eta,\\ \eta &= z_b + h\end{aligned}$$

Adhémar Jean-Claude Barré de Saint-Venant, 1871.

- \+ Simple
- \+ System of conservation laws
- \+ Fully non-linear
- \+ Numerically efficient
- − Vertical structure is entirely modelled
- − Hydrostatic, no-vertical acceleration/momentum
- Non-dispersive waves: $c = \sqrt{g\,h}$

1797–1886

$$
\begin{aligned}
\partial_t h + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}) &= 0,\\
\partial_t(h\,\mathbf{u}) + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}\,\mathbf{u}) &= -g\,h\,\boldsymbol{\nabla}\eta,\\
\eta &= z_b + h
\end{aligned}
$$

Adhémar Jean-Claude Barré de Saint-Venant, 1871.

+ Simple
+ System of conservation laws
+ Fully non-linear
+ Numerically efficient
– Vertical structure is entirely modelled
– Hydrostatic, no-vertical acceleration/momentum
Non-dispersive waves: $c = \sqrt{g\,h}$

1797–1886

$$
\begin{aligned}
\partial_t h + \boldsymbol{\nabla} \cdot (h\, \mathbf{u}) &= 0, \\
\partial_t (h\, \mathbf{u}) + \boldsymbol{\nabla} \cdot (h\, \mathbf{u}\, \mathbf{u}) &= -g\, h\, \boldsymbol{\nabla} \eta, \\
\eta &= z_b + h
\end{aligned}
$$

Adhémar Jean-Claude Barré de Saint-Venant, 1871.

- \+ Simple
- \+ System of conservation laws
- \+ Fully non-linear
- \+ Numerically efficient
- − Vertical structure is entirely modelled
- − Hydrostatic, no-vertical acceleration/momentum
- − Non-dispersive waves: $c = \sqrt{g\, h}$

1797–1886

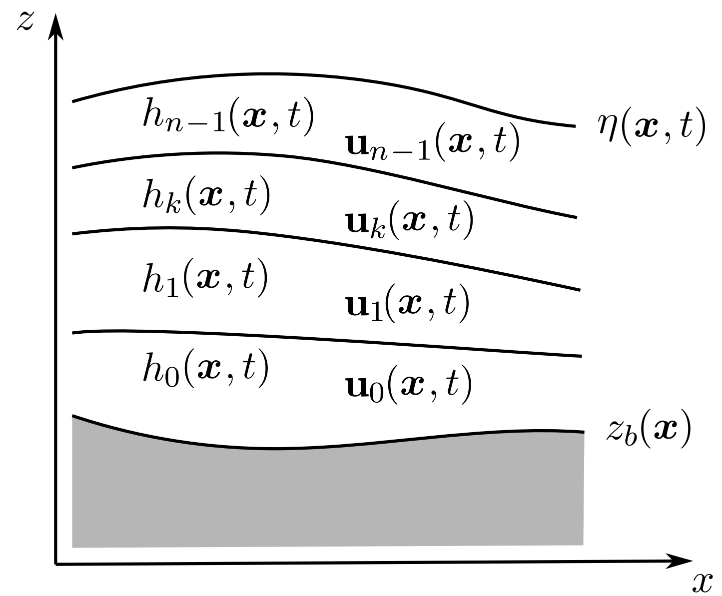
$$\begin{aligned} \partial_t h + \nabla \cdot (h\,\mathbf{u}) &= 0, \\ \partial_t (h\,\mathbf{u}) + \nabla \cdot (h\,\mathbf{u}\,\mathbf{u}) &= -g\,h\,\nabla\eta, \\ \eta &= z_b + h \end{aligned}$$

Adhémar Jean-Claude Barré de Saint-Venant, 1871.

- \+ Simple
- \+ System of conservation laws
- \+ Fully non-linear
- \+ Numerically efficient
- − Vertical structure is entirely modelled
- − Hydrostatic, no-vertical acceleration/momentum
- Non-dispersive waves: $c = \sqrt{g\,h}$

1797–1886

$$
\begin{aligned}
\partial_t h_k + \nabla \cdot (h\,\mathbf{u})_k &= 0, \\
\partial_t (h\,\mathbf{u})_k + \nabla \cdot (h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\nabla \eta, \\
\eta &= z_b + \sum h_k
\end{aligned}
$$

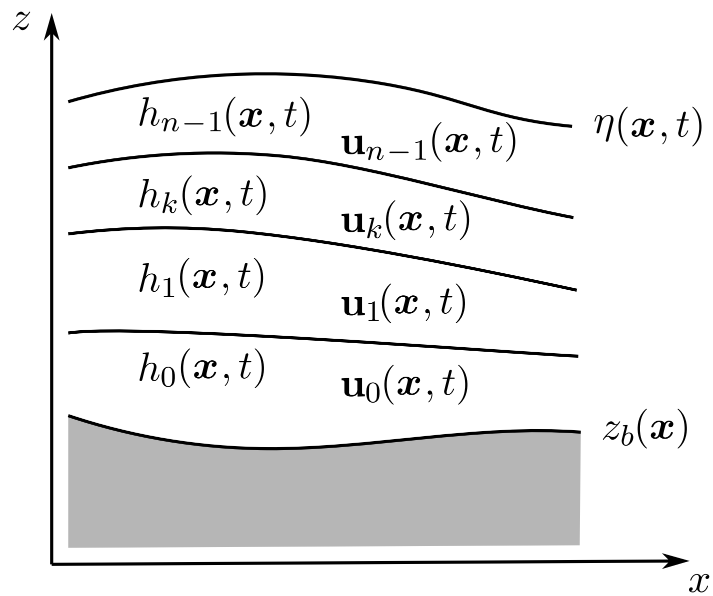


The basis for the "primitive equations" of oceanic and atmospheric circulation models.

Raymond B. Montgomery, 1937.

- \+ Vertical structure
- – Still hydrostatic / non-dispersive
- – Non-hyperbolic (i.e. multiple solutions)

$$\begin{aligned}\partial_t h_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u})_k &= 0,\\ \partial_t(h\,\mathbf{u})_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\boldsymbol{\nabla}\eta,\\ \eta &= z_b + \sum h_k\end{aligned}$$

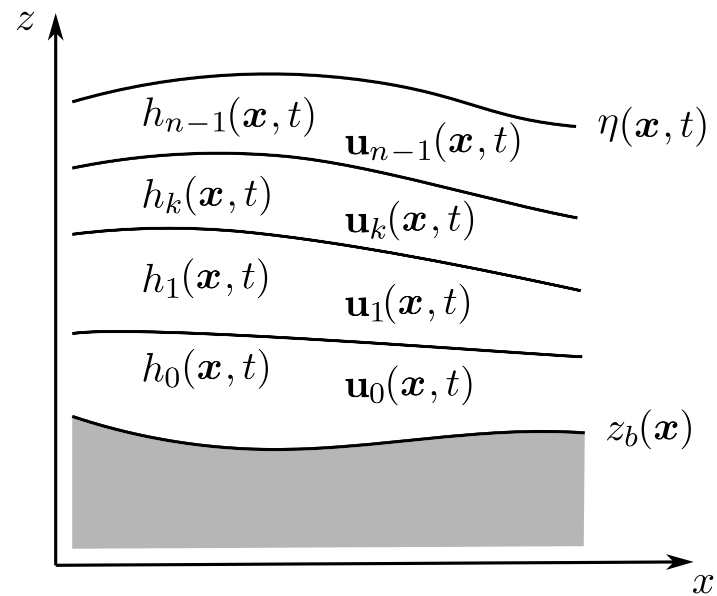


The basis for the "primitive equations" of oceanic and atmospheric circulation models.

Raymond B. Montgomery, 1937.

+ Vertical structure
– Still hydrostatic / non-dispersive
– Non-hyperbolic (i.e. multiple solutions)

$$
\begin{aligned}
\partial_t h_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u})_k &= 0,\\
\partial_t (h\,\mathbf{u})_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\boldsymbol{\nabla}\eta,\\
\eta &= z_b + \sum h_k
\end{aligned}
$$

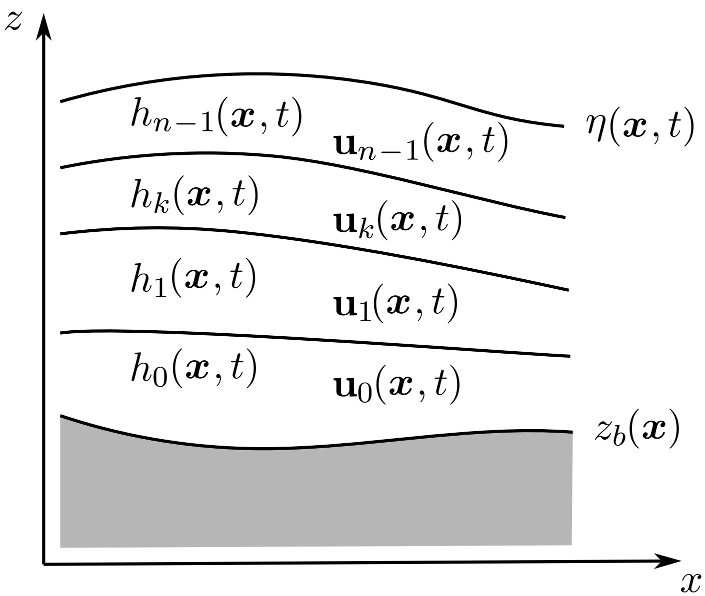


The basis for the "primitive equations" of oceanic and atmospheric circulation models.

Raymond B. Montgomery, 1937.

+ Vertical structure
– Still hydrostatic / non-dispersive
– Non-hyperbolic (i.e. multiple solutions)

$$\begin{aligned} \partial_t h_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u})_k &= 0, \\ \partial_t (h\,\mathbf{u})_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\boldsymbol{\nabla}\eta, \\ \eta &= z_b + \sum h_k \end{aligned}$$



The basis for the "primitive equations" of oceanic and atmospheric circulation models.

Raymond B. Montgomery, 1937.

+ Vertical structure
− Still hydrostatic / non-dispersive
− Non-hyperbolic (i.e. multiple solutions)

$$
\begin{aligned}
\partial_t h + \partial_x (h\, u) &= \frac{1}{6}\, H^3\, \frac{\partial^3 u}{\partial x^3}, \\
\partial_t u + u\, \partial_x u &= -g\, \partial_x \eta + \frac{1}{2}\, H^2\, \frac{\partial^3 u}{\partial t\, \partial x^2}
\end{aligned}
$$

Joseph Valentin Boussinesq, 1872 (also includes the "KdV" equation).
Many more recent variants: Serre 1953, Peregrine 1968, Green-Naghdi 1976...

\+ Non-hydrostatic, dispersive waves: $c^2 = g\, h\, \frac{1 + k^2 h^2/6}{1 + k^2 h^2/2}$

\+ Single layer

− Complex asymptotic terms

− High-order derivatives

− Not a system of conservation laws

− Vertical structure only for pressure

1842–1929

$$\partial_t h + \partial_x (h\, u) = \frac{1}{6} H^3 \frac{\partial^3 u}{\partial x^3},$$
$$\partial_t u + u\, \partial_x u = -g\, \partial_x \eta + \frac{1}{2} H^2 \frac{\partial^3 u}{\partial t\, \partial x^2}$$

Joseph Valentin Boussinesq, 1872 (also includes the "KdV" equation).
Many more recent variants: Serre 1953, Peregrine 1968, Green-Naghdi 1976…

+ Non-hydrostatic, dispersive waves: $c^2 = g\, h\, \frac{1 + k^2 h^2/6}{1 + k^2 h^2/2}$
+ Single layer
− Complex asymptotic terms
− High-order derivatives
− Not a system of conservation laws
− Vertical structure only for pressure

1842–1929

$$\partial_t h + \partial_x (h\,u) \;=\; \frac{1}{6}\,H^3\,\frac{\partial^3 u}{\partial x^3},$$
$$\partial_t u + u\,\partial_x u \;=\; -g\,\partial_x \eta + \frac{1}{2}\,H^2\,\frac{\partial^3 u}{\partial t\,\partial x^2}$$

Joseph Valentin Boussinesq, 1872 (also includes the "KdV" equation).
Many more recent variants: Serre 1953, Peregrine 1968, Green-Naghdi 1976...

+ Non-hydrostatic, dispersive waves: $c^2 = g\,h\,\frac{1+k^2\,h^2/6}{1+k^2\,h^2/2}$
+ Single layer
− Complex asymptotic terms
− High-order derivatives
− Not a system of conservation laws
− Vertical structure only for pressure

1842–1929

$$
\begin{aligned}
\partial_t h + \partial_x (h\,u) &= \frac{1}{6}\, H^3\, \frac{\partial^3 u}{\partial x^3}, \\
\partial_t u + u\,\partial_x u &= -g\,\partial_x \eta + \frac{1}{2}\, H^2\, \frac{\partial^3 u}{\partial t\,\partial x^2}
\end{aligned}
$$

Joseph Valentin Boussinesq, 1872 (also includes the "KdV" equation).
Many more recent variants: Serre 1953, Peregrine 1968, Green-Naghdi 1976...

- \+ Non-hydrostatic, dispersive waves: $c^2 = g\,h\,\frac{1+k^2\,h^2/6}{1+k^2\,h^2/2}$
- \+ Single layer
- − Complex asymptotic terms
- − High-order derivatives
- − Not a system of conservation laws
- − Vertical structure only for pressure

1842–1929

$$\partial_t h + \partial_x (h\,u) = \frac{1}{6}\,H^3\,\frac{\partial^3 u}{\partial x^3},$$
$$\partial_t u + u\,\partial_x u = -g\,\partial_x \eta + \frac{1}{2}\,H^2\,\frac{\partial^3 u}{\partial t\,\partial x^2}$$

Joseph Valentin Boussinesq, 1872 (also includes the "KdV" equation).
Many more recent variants: Serre 1953, Peregrine 1968, Green-Naghdi 1976...

- \+ Non-hydrostatic, dispersive waves: $c^2 = g\,h\,\frac{1+k^2\,h^2/6}{1+k^2\,h^2/2}$
- \+ Single layer
- − Complex asymptotic terms
- − High-order derivatives
- − Not a system of conservation laws
- − Vertical structure only for pressure

1842–1929

$$\begin{aligned}\partial_t h + \partial_x (h\,u) &= \frac{1}{6} H^3 \frac{\partial^3 u}{\partial x^3},\\ \partial_t u + u\,\partial_x u &= -g\,\partial_x \eta + \frac{1}{2} H^2 \frac{\partial^3 u}{\partial t\,\partial x^2}\end{aligned}$$

Joseph Valentin Boussinesq, 1872 (also includes the "KdV" equation).
Many more recent variants: Serre 1953, Peregrine 1968, Green-Naghdi 1976...

- \+ Non-hydrostatic, dispersive waves: $c^2 = g\,h\,\frac{1+k^2\,h^2/6}{1+k^2\,h^2/2}$
- \+ Single layer
- – Complex asymptotic terms
- – High-order derivatives
- – Not a system of conservation laws
- – Vertical structure only for pressure

1842–1929

$$\begin{aligned} \partial_t h + \partial_x (h\,u) &= \frac{1}{6}\,H^3\,\frac{\partial^3 u}{\partial x^3}, \\ \partial_t u + u\,\partial_x u &= -g\,\partial_x \eta + \frac{1}{2}\,H^2\,\frac{\partial^3 u}{\partial t\,\partial x^2} \end{aligned}$$

Joseph Valentin Boussinesq, 1872 (also includes the "KdV" equation).
Many more recent variants: Serre 1953, Peregrine 1968, Green-Naghdi 1976...

+ Non-hydrostatic, dispersive waves: $c^2 = g\,h\,\frac{1+k^2\,h^2/6}{1+k^2\,h^2/2}$
+ Single layer
− Complex asymptotic terms
− High-order derivatives
− Not a system of conservation laws
− Vertical structure only for pressure

1842–1929

$$\begin{aligned} \partial_t \mathbf{u} + \boldsymbol{\nabla} \cdot (\mathbf{u} \otimes \mathbf{u}) &= -\frac{1}{\rho} \nabla p + \mathbf{g} \\ \boldsymbol{\nabla} \cdot \mathbf{u} &= 0 \\ \partial_t \boldsymbol{\chi}(\boldsymbol{x}, t) &= \mathbf{u}(\boldsymbol{\chi}(\boldsymbol{x}, t), t) \end{aligned}$$

Leonhard Euler, 1757 (without the free surface).

+ Complete/consistent system
– Complex boundary condition
– Link with Saint-Venant?
– How to represent χ? (VOF, Levelset, Lagrangian etc.)
– Computational cost

1707-1783

$$\begin{aligned}\partial_t \mathbf{u} + \boldsymbol{\nabla}\cdot(\mathbf{u}\otimes\mathbf{u}) &= -\frac{1}{\rho}\nabla p + \mathbf{g}\\ \boldsymbol{\nabla}\cdot\mathbf{u} &= 0\\ \partial_t \boldsymbol{\chi}(\boldsymbol{x},t) &= \mathbf{u}(\boldsymbol{\chi}(\boldsymbol{x},t),t)\end{aligned}$$

Leonhard Euler, 1757 (without the free surface).

+ Complete/consistent system
− Complex boundary condition
− Link with Saint-Venant?
− How to represent χ? (VOF, Levelset, Lagrangian etc.)
− Computational cost

1707-1783

$$
\begin{aligned}
\partial_t \mathbf{u} + \boldsymbol{\nabla} \cdot (\mathbf{u} \otimes \mathbf{u}) &= -\frac{1}{\rho} \nabla p + \mathbf{g} \\
\boldsymbol{\nabla} \cdot \mathbf{u} &= 0 \\
\partial_t \boldsymbol{\chi}(\boldsymbol{x}, t) &= \mathbf{u}(\boldsymbol{\chi}(\boldsymbol{x}, t), t)
\end{aligned}
$$

Leonhard Euler, 1757 (without the free surface).

+ Complete/consistent system
– Complex boundary condition
– Link with Saint-Venant?
– How to represent χ? (VOF, Levelset, Lagrangian etc.)
– Computational cost

1707-1783

$$
\begin{aligned}
\partial_t \mathbf{u} + \boldsymbol{\nabla} \cdot (\mathbf{u} \otimes \mathbf{u}) &= -\frac{1}{\rho} \nabla p + \mathbf{g} \\
\boldsymbol{\nabla} \cdot \mathbf{u} &= 0 \\
\partial_t \boldsymbol{\chi}(\boldsymbol{x}, t) &= \mathbf{u}(\boldsymbol{\chi}(\boldsymbol{x}, t), t)
\end{aligned}
$$

Leonhard Euler, 1757 (without the free surface).

+ Complete/consistent system
− Complex boundary condition
− Link with Saint-Venant?
− How to represent χ? (VOF, Levelset, Lagrangian etc.)
− Computational cost

1707-1783

$$
\begin{aligned}
\partial_t \mathbf{u} + \boldsymbol{\nabla} \cdot (\mathbf{u} \otimes \mathbf{u}) &= -\frac{1}{\rho} \nabla p + \mathbf{g} \\
\boldsymbol{\nabla} \cdot \mathbf{u} &= 0 \\
\partial_t \boldsymbol{\chi}(\boldsymbol{x}, t) &= \mathbf{u}(\boldsymbol{\chi}(\boldsymbol{x}, t), t)
\end{aligned}
$$

Leonhard Euler, 1757 (without the free surface).

1707-1783

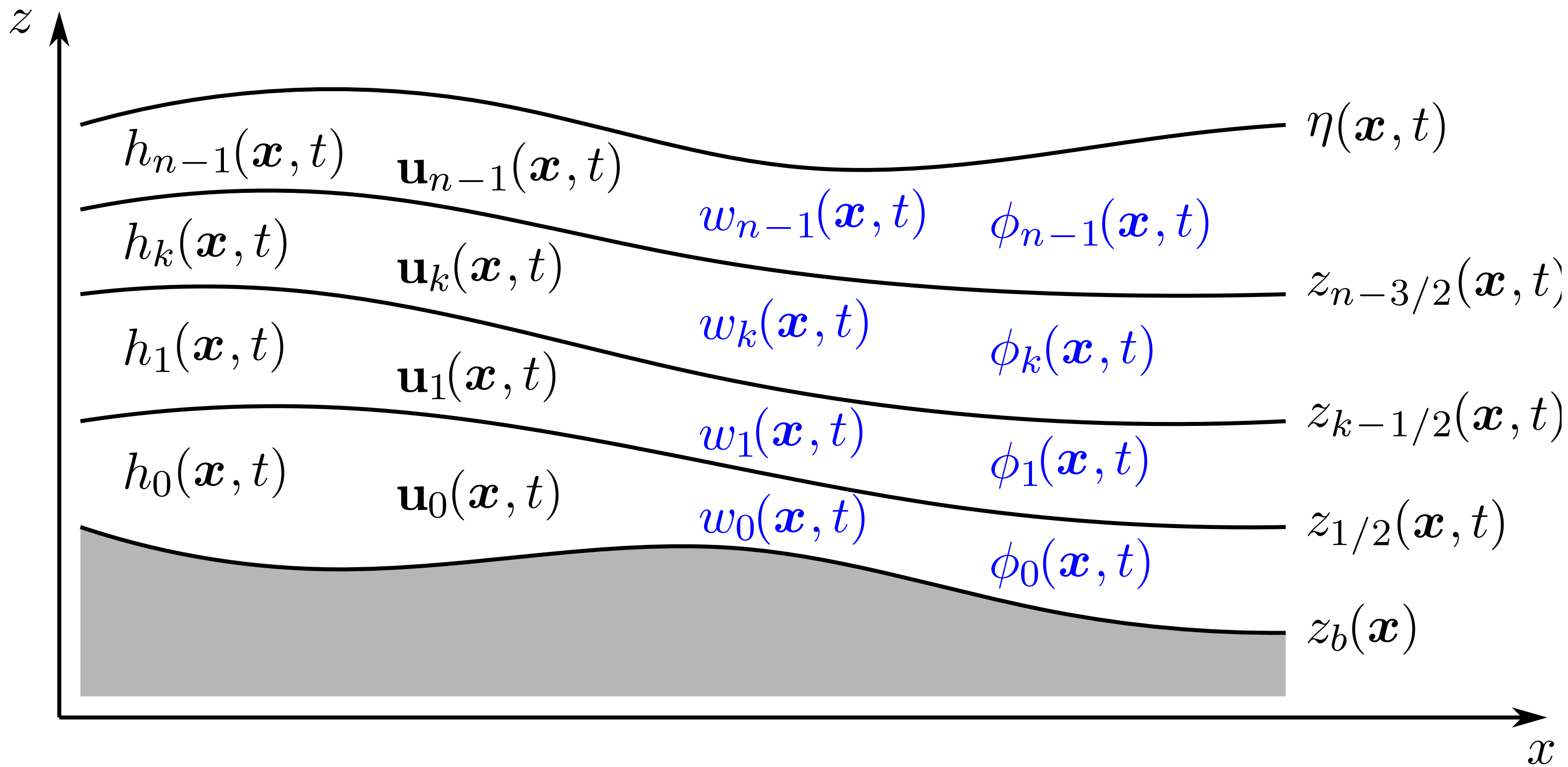
+ Complete/consistent system
− Complex boundary condition
− Link with Saint-Venant?
− How to represent χ? (VOF, Levelset, Lagrangian etc.)
− Computational cost

$$\begin{aligned}\partial_t \mathbf{u} + \boldsymbol{\nabla}\cdot(\mathbf{u}\otimes\mathbf{u}) &= -\frac{1}{\rho}\nabla p + \mathbf{g}\\ \boldsymbol{\nabla}\cdot\mathbf{u} &= 0\\ \partial_t \boldsymbol{\chi}(\boldsymbol{x},t) &= \mathbf{u}(\boldsymbol{\chi}(\boldsymbol{x},t),t)\end{aligned}$$

Leonhard Euler, 1757 (without the free surface).

+ Complete/consistent system
− Complex boundary condition
− Link with Saint-Venant?
− How to represent χ? (VOF, Levelset, Lagrangian etc.)
− Computational cost

1707-1783



$\phi = \frac{p_{\rm nh}}{\rho}$ with $p_{\rm nh}$ the non-hydrostatic pressure.

$$
\begin{aligned}
\partial_t h_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u})_k &= 0,\\
\partial_t (h\,\mathbf{u})_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\boldsymbol{\nabla}\eta - \boldsymbol{\nabla}(h\,\phi)_k + [\phi\,\boldsymbol{\nabla}z]_k,\\
\partial_t (h\,w)_k + \boldsymbol{\nabla}\cdot(h\,w\,\mathbf{u})_k &= -[\phi]_k,\\
\boldsymbol{\nabla}\cdot(h\,\mathbf{u})_k + [w - \mathbf{u}\cdot\boldsymbol{\nabla}z]_k &= 0,\\
[f]_k &= f_{k+1/2} - f_{k-1/2},
\end{aligned}
$$

- \+ Consistent discretisation of the full 3D Euler equations
- \+ Clearly connected with the simpler hydrostatic system
- \+ System of (discrete) conservation laws
- \+ Numerically efficient
- +– Vertically Lagrangian
- – Too many unknowns!

$$\begin{aligned}
\partial_t h_k + \nabla\cdot(h\,\mathbf{u})_k &= 0,\\
\partial_t (h\,\mathbf{u})_k + \nabla\cdot(h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\nabla\eta - \nabla(h\,\phi)_k + [\phi\,\nabla z]_k,\\
\partial_t (h\,w)_k + \nabla\cdot(h\,w\,\mathbf{u})_k &= -[\phi]_k,\\
\nabla\cdot(h\,\mathbf{u})_k + [w - \mathbf{u}\cdot\nabla z]_k &= 0,\\
[f]_k &= f_{k+1/2} - f_{k-1/2},
\end{aligned}$$

- \+ Consistent discretisation of the full 3D Euler equations
- \+ Clearly connected with the simpler hydrostatic system
- \+ System of (discrete) conservation laws
- \+ Numerically efficient
- +− Vertically Lagrangian
- − Too many unknowns!

$$
\begin{aligned}
\partial_t h_k + \nabla \cdot (h\,\mathbf{u})_k &= 0, \\
\partial_t (h\,\mathbf{u})_k + \nabla \cdot (h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\nabla \eta - \nabla (h\,\phi)_k + [\phi\,\nabla z]_k, \\
\partial_t (h\,w)_k + \nabla \cdot (h\,w\,\mathbf{u})_k &= -[\phi]_k, \\
\nabla \cdot (h\,\mathbf{u})_k + [w - \mathbf{u} \cdot \nabla z]_k &= 0, \\
[f]_k &= f_{k+1/2} - f_{k-1/2},
\end{aligned}
$$

- \+ Consistent discretisation of the full 3D Euler equations
- \+ Clearly connected with the simpler hydrostatic system
- \+ System of (discrete) conservation laws
- \+ Numerically efficient
- +− Vertically Lagrangian
- − Too many unknowns!

$$
\begin{aligned}
\partial_t h_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u})_k &= 0,\\
\partial_t (h\,\mathbf{u})_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\boldsymbol{\nabla}\eta - \boldsymbol{\nabla}(h\,\phi)_k + [\phi\,\boldsymbol{\nabla}z]_k,\\
\partial_t (h\,w)_k + \boldsymbol{\nabla}\cdot(h\,w\,\mathbf{u})_k &= -[\phi]_k,\\
\boldsymbol{\nabla}\cdot(h\,\mathbf{u})_k + [w - \mathbf{u}\cdot\boldsymbol{\nabla}z]_k &= 0,\\
[f]_k &= f_{k+1/2} - f_{k-1/2},
\end{aligned}
$$

+ Consistent discretisation of the full 3D Euler equations
+ Clearly connected with the simpler hydrostatic system
+ System of (discrete) conservation laws
+ Numerically efficient
+− Vertically Lagrangian
− Too many unknowns!

$$
\begin{aligned}
\partial_t h_k + \boldsymbol{\nabla} \cdot (h\,\mathbf{u})_k &= 0, \\
\partial_t (h\,\mathbf{u})_k + \boldsymbol{\nabla} \cdot (h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\boldsymbol{\nabla}\eta - \boldsymbol{\nabla}(h\,\phi)_k + [\phi\,\boldsymbol{\nabla} z]_k, \\
\partial_t (h\,w)_k + \boldsymbol{\nabla} \cdot (h\,w\,\mathbf{u})_k &= -[\phi]_k, \\
\boldsymbol{\nabla} \cdot (h\,\mathbf{u})_k + [w - \mathbf{u} \cdot \boldsymbol{\nabla} z]_k &= 0, \\
[f]_k &= f_{k+1/2} - f_{k-1/2},
\end{aligned}
$$

- \+ Consistent discretisation of the full 3D Euler equations
- \+ Clearly connected with the simpler hydrostatic system
- \+ System of (discrete) conservation laws
- \+ Numerically efficient
- \+ − Vertically Lagrangian
- − Too many unknowns!

$$
\begin{aligned}
\partial_t h_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u})_k &= 0,\\
\partial_t (h\,\mathbf{u})_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\boldsymbol{\nabla}\eta - \boldsymbol{\nabla}(h\,\phi)_k + [\phi\,\boldsymbol{\nabla} z]_k,\\
\partial_t (h\,w)_k + \boldsymbol{\nabla}\cdot(h\,w\,\mathbf{u})_k &= -[\phi]_k,\\
\boldsymbol{\nabla}\cdot(h\,\mathbf{u})_k + [w - \mathbf{u}\cdot\boldsymbol{\nabla} z]_k &= 0,\\
[f]_k &= f_{k+1/2} - f_{k-1/2},
\end{aligned}
$$

- \+ Consistent discretisation of the full 3D Euler equations
- \+ Clearly connected with the simpler hydrostatic system
- \+ System of (discrete) conservation laws
- \+ Numerically efficient
- +− Vertically Lagrangian
- − Too many unknowns!

$$
\begin{aligned}
\partial_t h_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u})_k &= 0,\\
\partial_t (h\,\mathbf{u})_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\boldsymbol{\nabla}\eta - \boldsymbol{\nabla}(h\,\phi)_k + [\phi\,\boldsymbol{\nabla} z]_k,\\
\partial_t (h\,w)_k + \boldsymbol{\nabla}\cdot(h\,w\,\mathbf{u})_k &= -[\phi]_k,\\
\boldsymbol{\nabla}\cdot(h\,\mathbf{u})_k + [w - \mathbf{u}\cdot\boldsymbol{\nabla} z]_k &= 0,\\
[f]_k &= f_{k+1/2} - f_{k-1/2},
\end{aligned}
$$

+ Consistent discretisation of the full 3D Euler equations
+ Clearly connected with the simpler hydrostatic system
+ System of (discrete) conservation laws
+ Numerically efficient
+− Vertically Lagrangian
− Too many unknowns!

$$
\begin{aligned}
\partial_t h_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u})_k &= 0,\\
\partial_t (h\,\mathbf{u})_k + \boldsymbol{\nabla}\cdot(h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\boldsymbol{\nabla}\eta - \boldsymbol{\nabla}(h\,\phi)_k + [\phi\,\boldsymbol{\nabla} z]_k,\\
\partial_t (h\,w)_k + \boldsymbol{\nabla}\cdot(h\,w\,\mathbf{u})_k &= -[\phi]_k,\\
\boldsymbol{\nabla}\cdot(h\,\mathbf{u})_k + [w - \mathbf{u}\cdot\boldsymbol{\nabla} z]_k &= 0,\\
[f]_k &= f_{k+1/2} - f_{k-1/2},
\end{aligned}
$$

+ Consistent discretisation of the full 3D Euler equations
+ Clearly connected with the simpler hydrostatic system
+ System of (discrete) conservation laws
+ Numerically efficient

+− Vertically Lagrangian

− Too many unknowns!

$$\begin{aligned}
\partial_t h_k + \boldsymbol{\nabla} \cdot (h\,\mathbf{u})_k &= 0,\\
\partial_t (h\,\mathbf{u})_k + \boldsymbol{\nabla} \cdot (h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\boldsymbol{\nabla}\eta - \boldsymbol{\nabla}(h\,\phi)_k + [\phi\,\boldsymbol{\nabla} z]_k,\\
\partial_t (h\,w)_k + \boldsymbol{\nabla} \cdot (h\,w\,\mathbf{u})_k &= -[\phi]_k,\\
\boldsymbol{\nabla} \cdot (h\,\mathbf{u})_k + [w - \mathbf{u}\cdot\boldsymbol{\nabla} z]_k &= 0,\\
[f]_k &= f_{k+1/2} - f_{k-1/2},
\end{aligned}$$

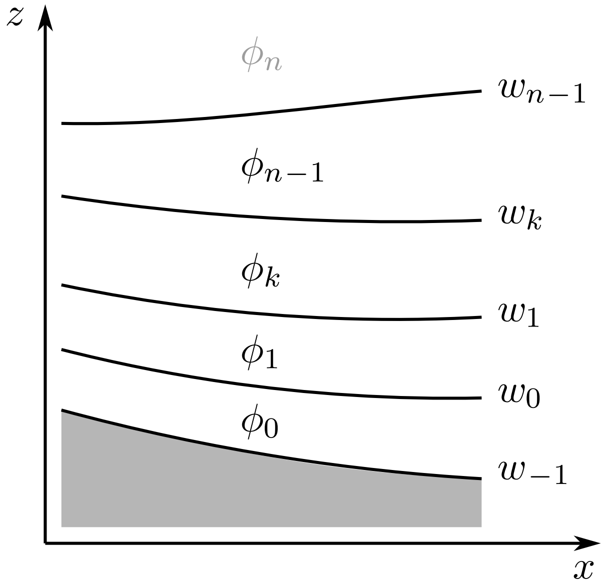
- \+ Consistent discretisation of the full 3D Euler equations
- \+ Clearly connected with the simpler hydrostatic system
- \+ System of (discrete) conservation laws
- \+ Numerically efficient
- +− Vertically Lagrangian
- − Too many unknowns!

$$
\begin{aligned}
\partial_t h_k + \nabla \cdot (h\,\mathbf{u})_k &= 0, \\
\partial_t (h\,\mathbf{u})_k + \nabla \cdot (h\,\mathbf{u}\,\mathbf{u})_k &= -g\,h_k\,\nabla \eta - \nabla (h\,\phi)_k + [\phi\,\nabla z]_k, \\
\partial_t (h\,w)_k + \nabla \cdot (h\,w\,\mathbf{u})_k &= -[\phi]_k, \\
\nabla \cdot (h\,\mathbf{u})_k + [w - \mathbf{u} \cdot \nabla z]_k &= 0, \\
[f]_k &= f_{k+1/2} - f_{k-1/2},
\end{aligned}
$$

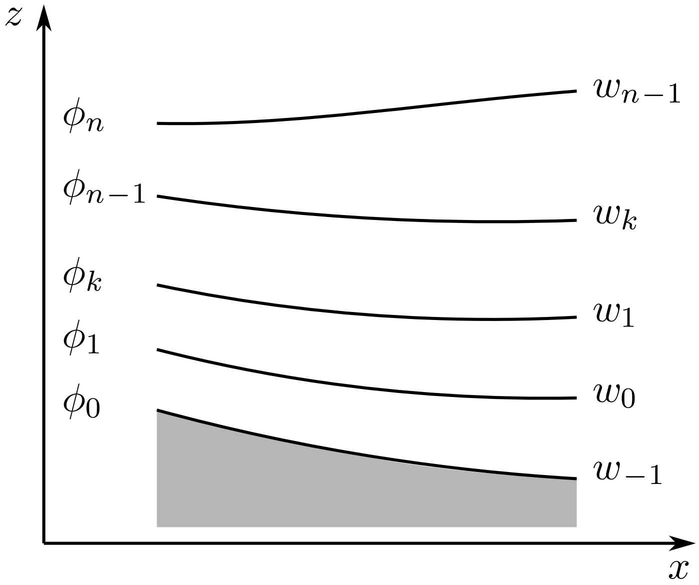
- \+ Consistent discretisation of the full 3D Euler equations
- \+ Clearly connected with the simpler hydrostatic system
- \+ System of (discrete) conservation laws
- \+ Numerically efficient
- +– Vertically Lagrangian
- – Too many unknowns!

Examples of possible vertical discretisations

1917–2008



Lorenz grid



Keller grid (box scheme)

Edward Norton Lorenz, *Energy and numerical weather prediction*, 1960.

H. B. Keller, *A new difference scheme for parabolic problems*. 1971.

Zijlema & Stelling, IJNMF, 2005.

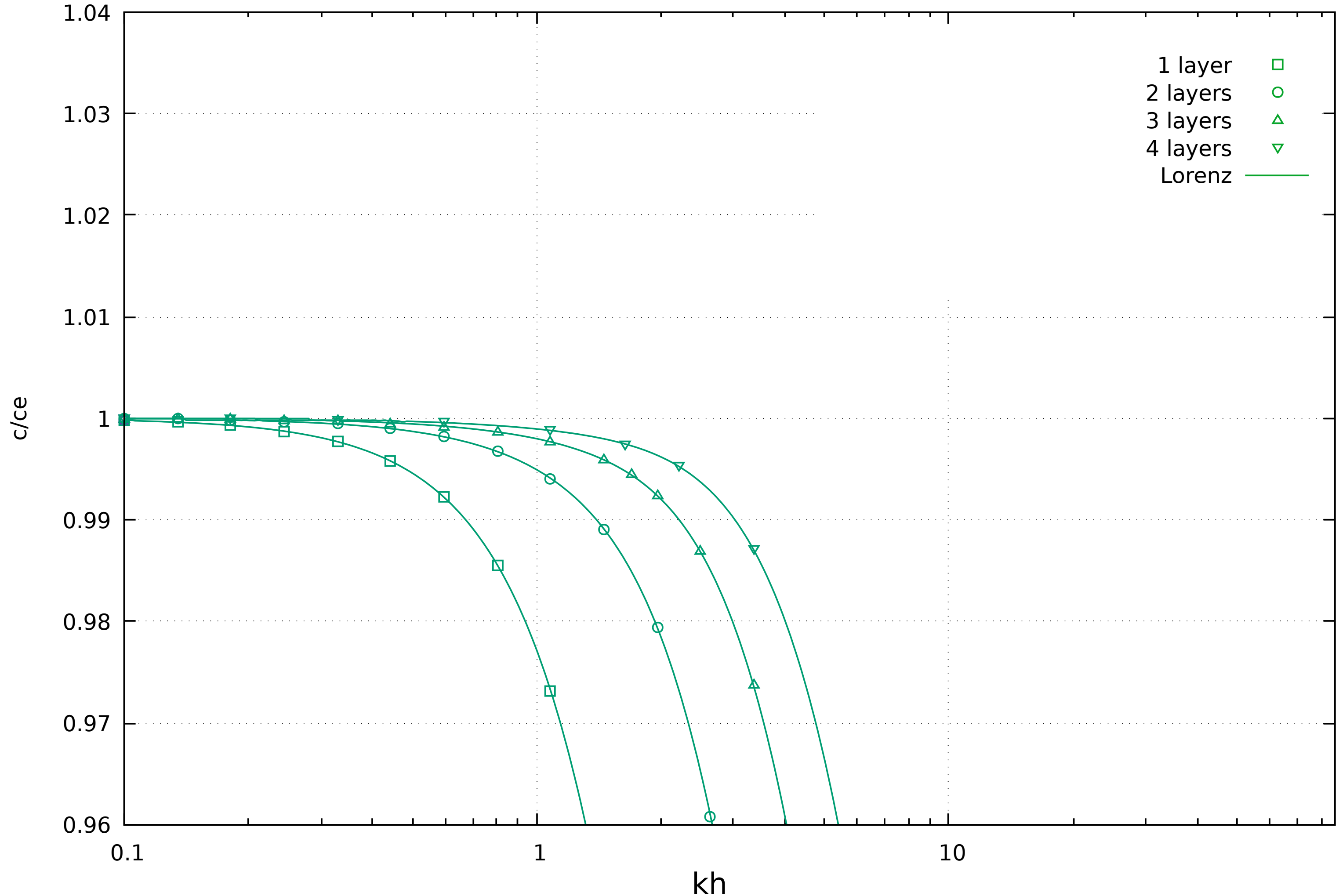
Which one is best? $\rightarrow$ choose the best dispersion relation based on the *linearised perturbation system* ($h_k = \bar{h}_k + h'_k\, e^{i(\hat{k}x - \omega t)}$, $u_k = u'_k\, e^{i(\hat{k}x - \omega t)}$, etc.)

$$
\begin{aligned}
-\omega\, h'_k + \bar{h}_k\, u'_k\, \hat{k} &= 0, \\
-\bar{h}_k\, u'_k\, \omega &= -g\, \hat{k}\, \bar{h}_k \sum h'_k - \bar{h}_k\, \hat{k}\, \frac{\phi'_{k+1} + \phi'_k}{2}, \\
-\bar{h}_k\, \frac{w'_k + w'_{k-1}}{2}\, \omega\, i &= \phi'_k - \phi'_{k+1}, \\
\bar{h}_k\, u'_k\, \hat{k}\, i + w'_k - w'_{k-1} &= 0,
\end{aligned}
$$

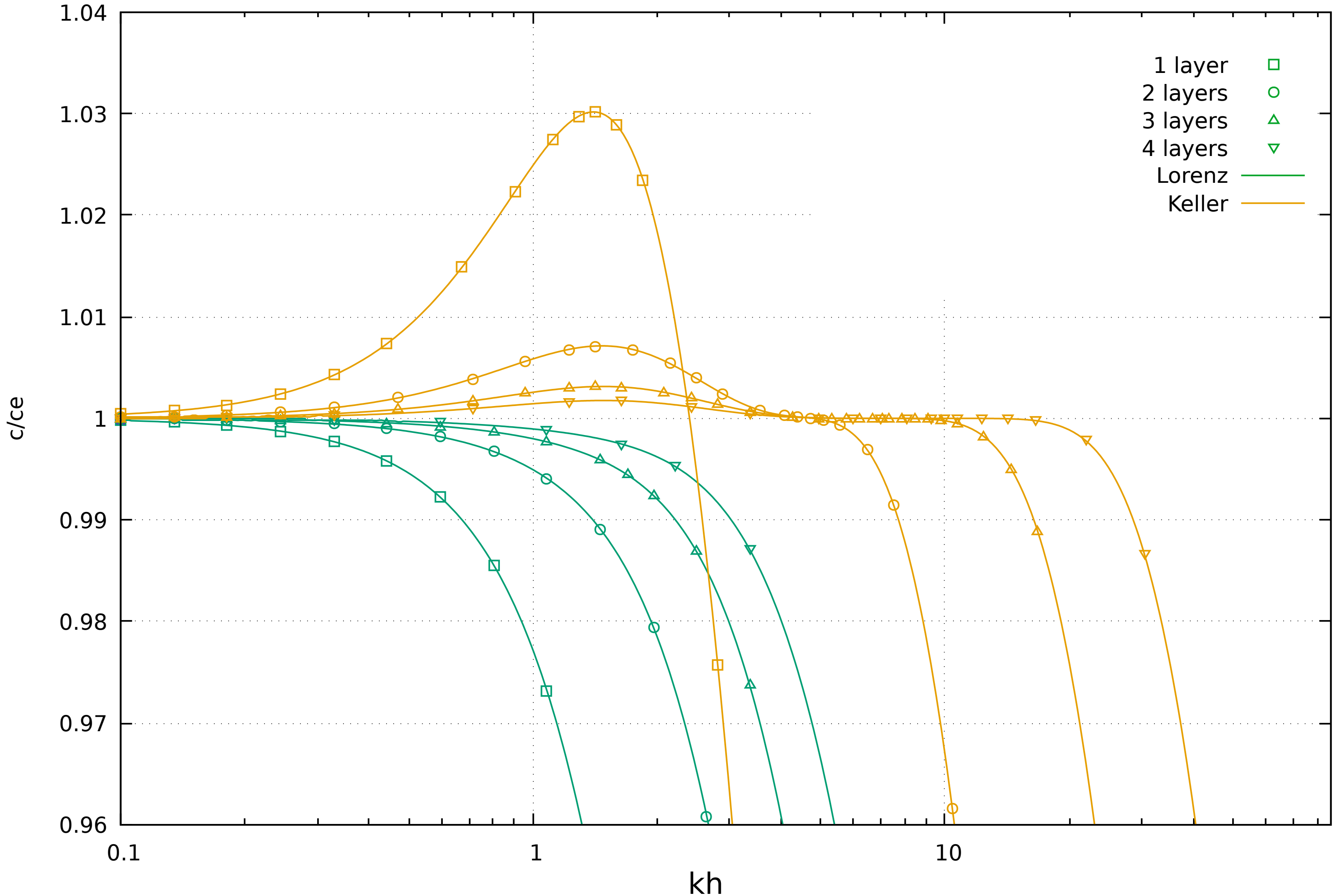
Computing the determinant then gives (for two layers)

$$
\omega_2^2(\hat{k}) = 8\, g\, \frac{h_0\, h_1^2\, \hat{k}^4 + (h_1 + h_0)\, \hat{k}^2}{3\, h_0^2\, h_1^2\, \hat{k}^4 + 3\,(3\, h_0^2 + h_0\, h_1 + h_1^2)\, \hat{k}^2 + 8}
$$

Dispersion relations

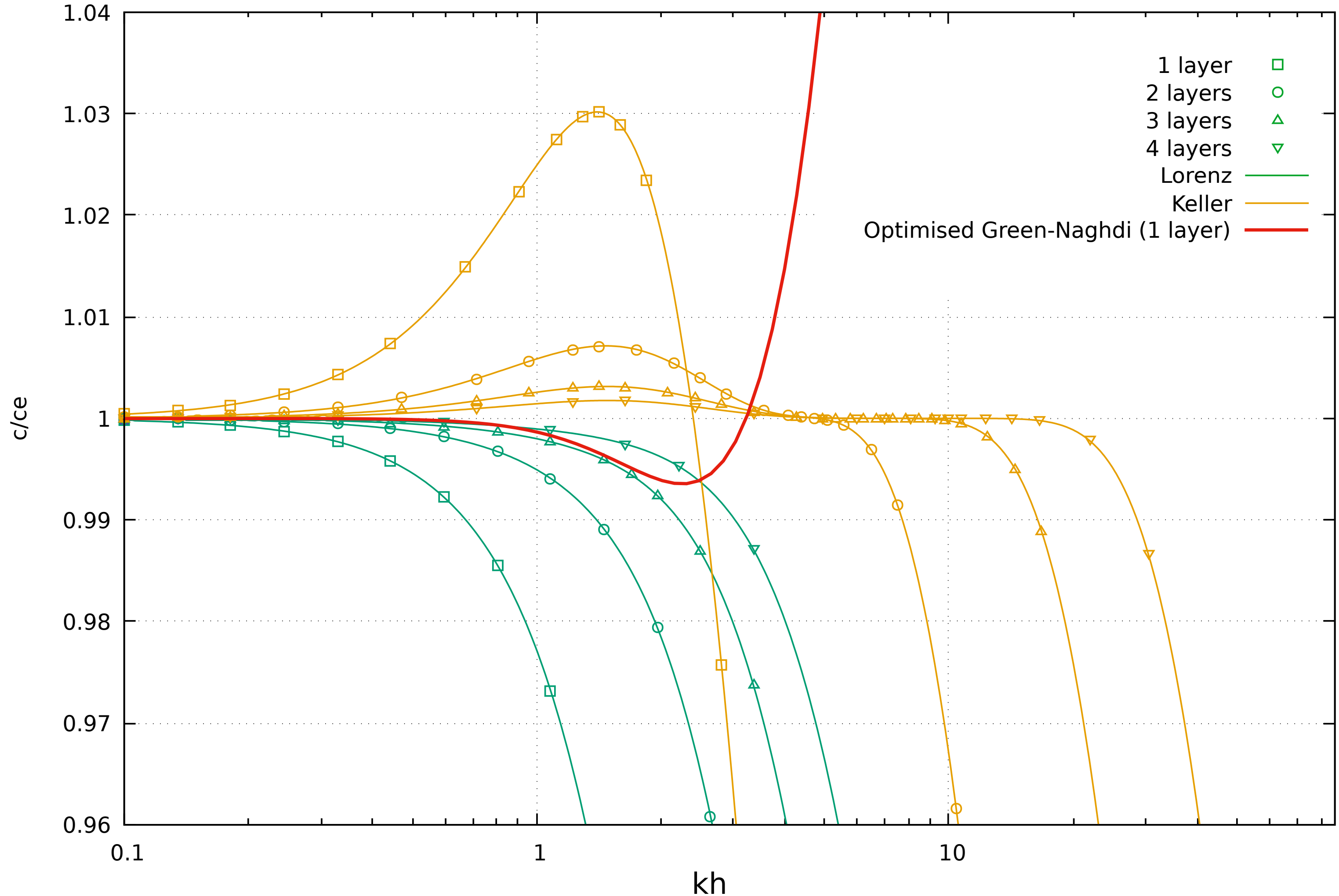


$$c_e^2 = \frac{g}{k}\tanh(k\,h)$$

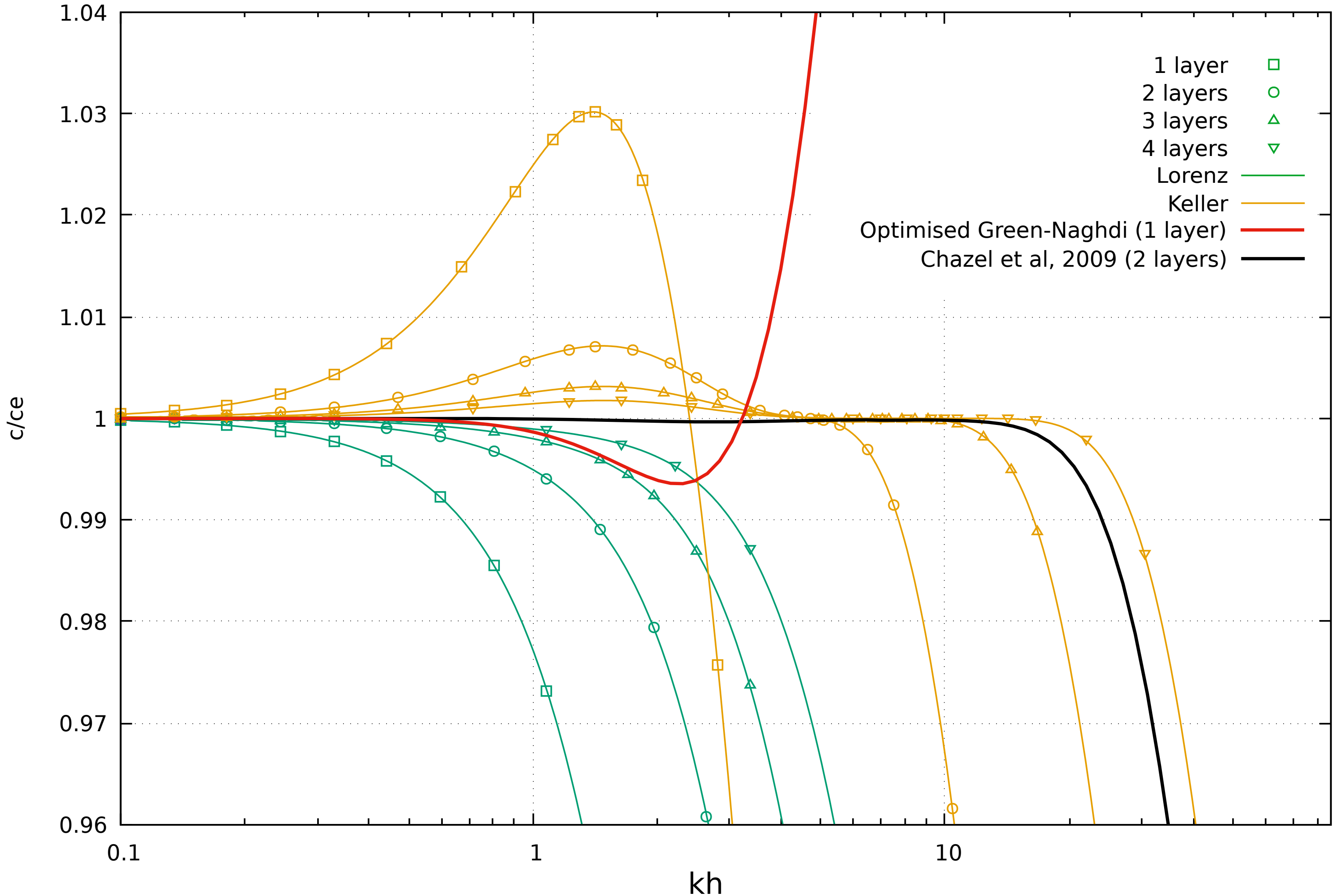


$$c_e^2 = \frac{g}{k} \tanh(k\,h)$$

Dispersion relations

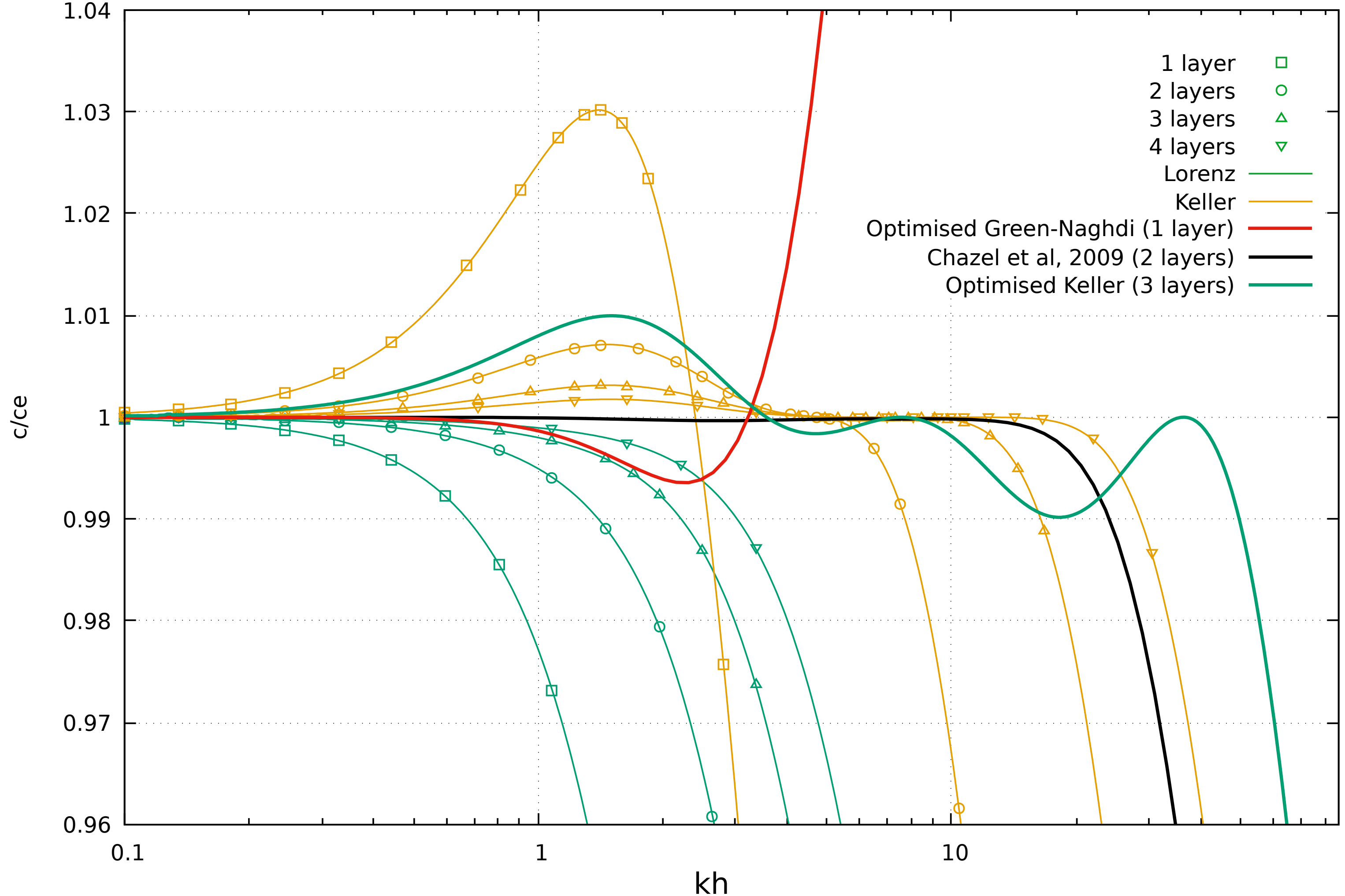


$$c_e^2 = \frac{g}{k}\tanh(k\,h)$$



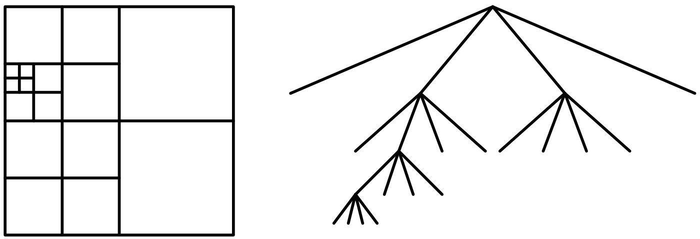
$$c_e^2 = \frac{g}{k}\tanh(k\,h)$$

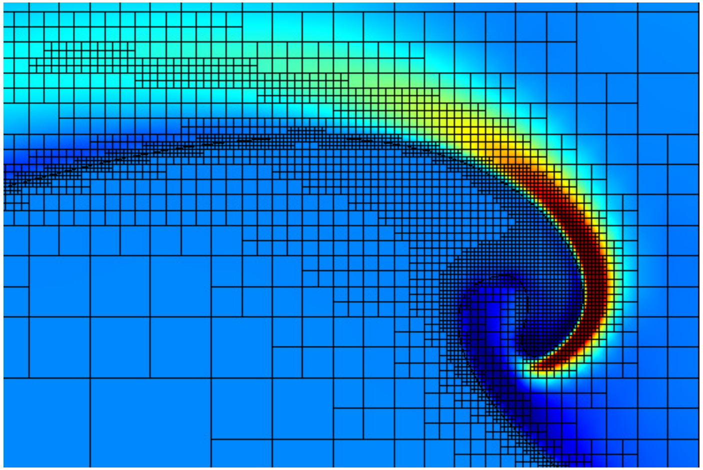
Dispersion relations

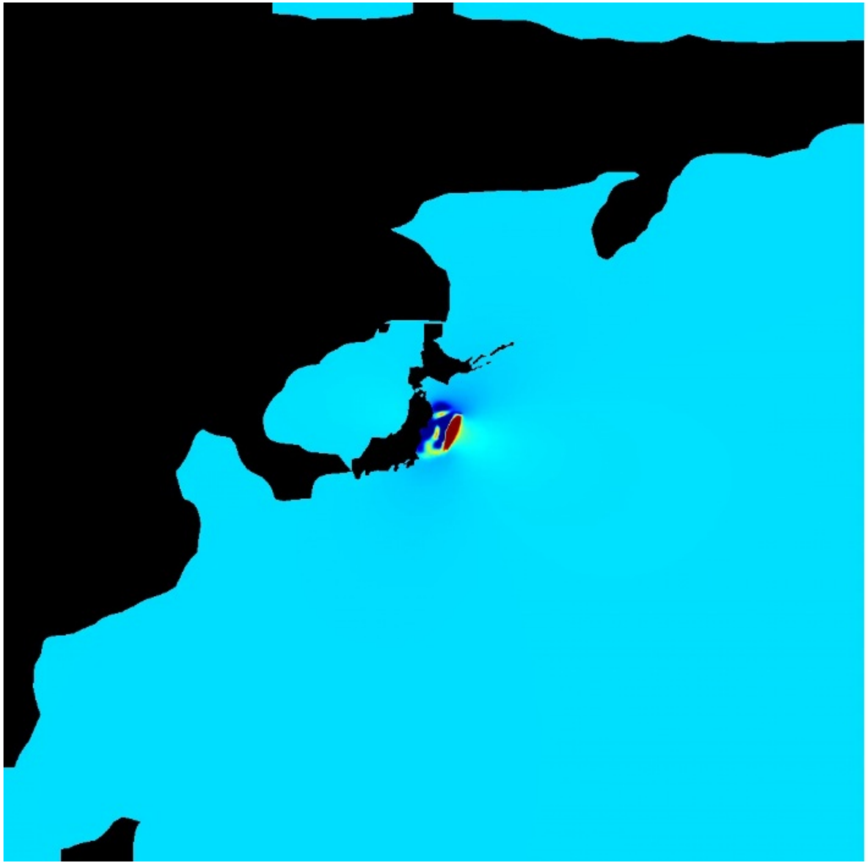


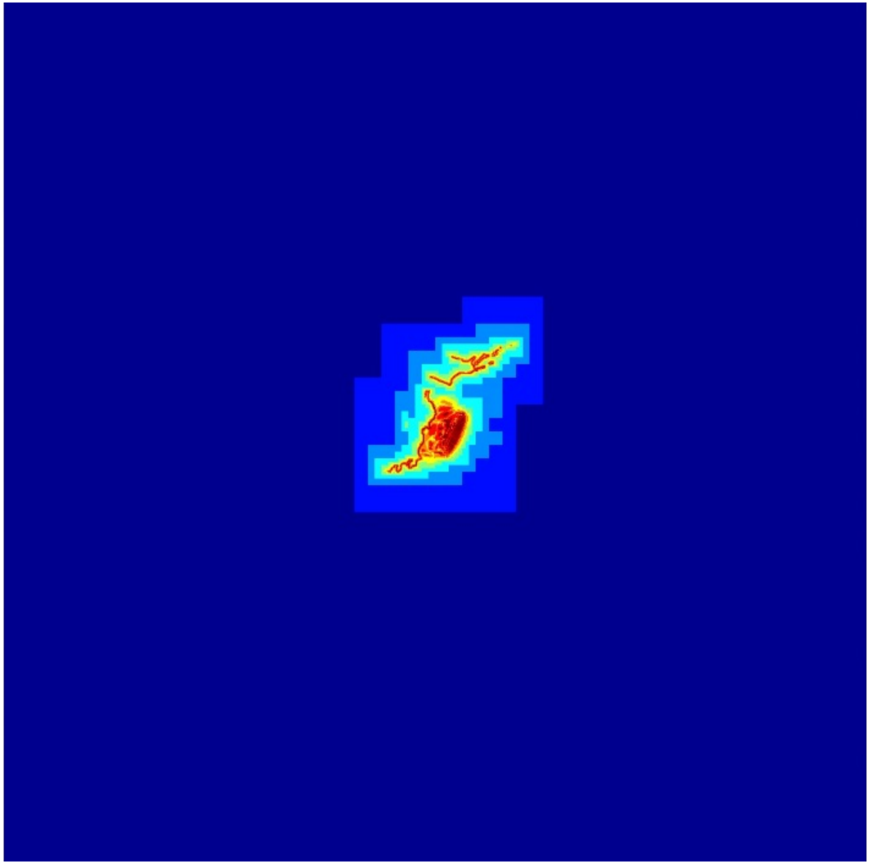
$$c_e^2 = \frac{g}{k}\tanh(k\,h)$$

- Basilisk: open-source, `basilisk.fr`
- Quad/octree adaptive mesh refinement
- Multigrid solver with (vertical) line relaxations for elliptic problems
- MPI/Open MP parallelism
- Load-balancing based on space-filling curves (Z-ordering) (scales to 1000s of cores)
- Decoupled (high-level) numerical scheme / (low-level) memory and loop traversal implementation
- Documented, reproducible and editable examples and test cases: "wikipedia of fluid mechanics"







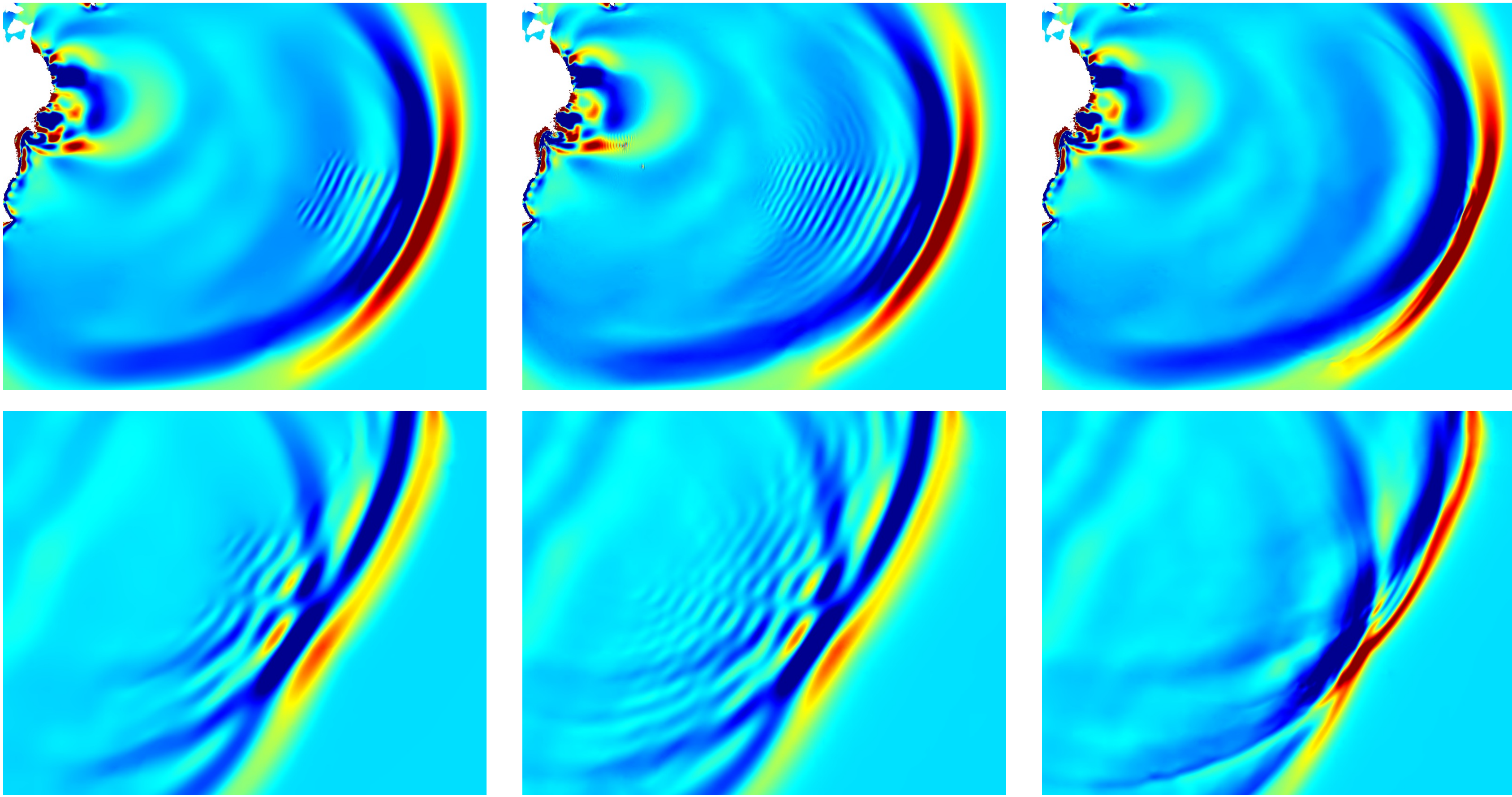


6000×8000 km, spatial resolution: 1–250 km

See also Popinet, 2015 (Green-Naghdi) and Popinet, 2012 (Saint-Venant)

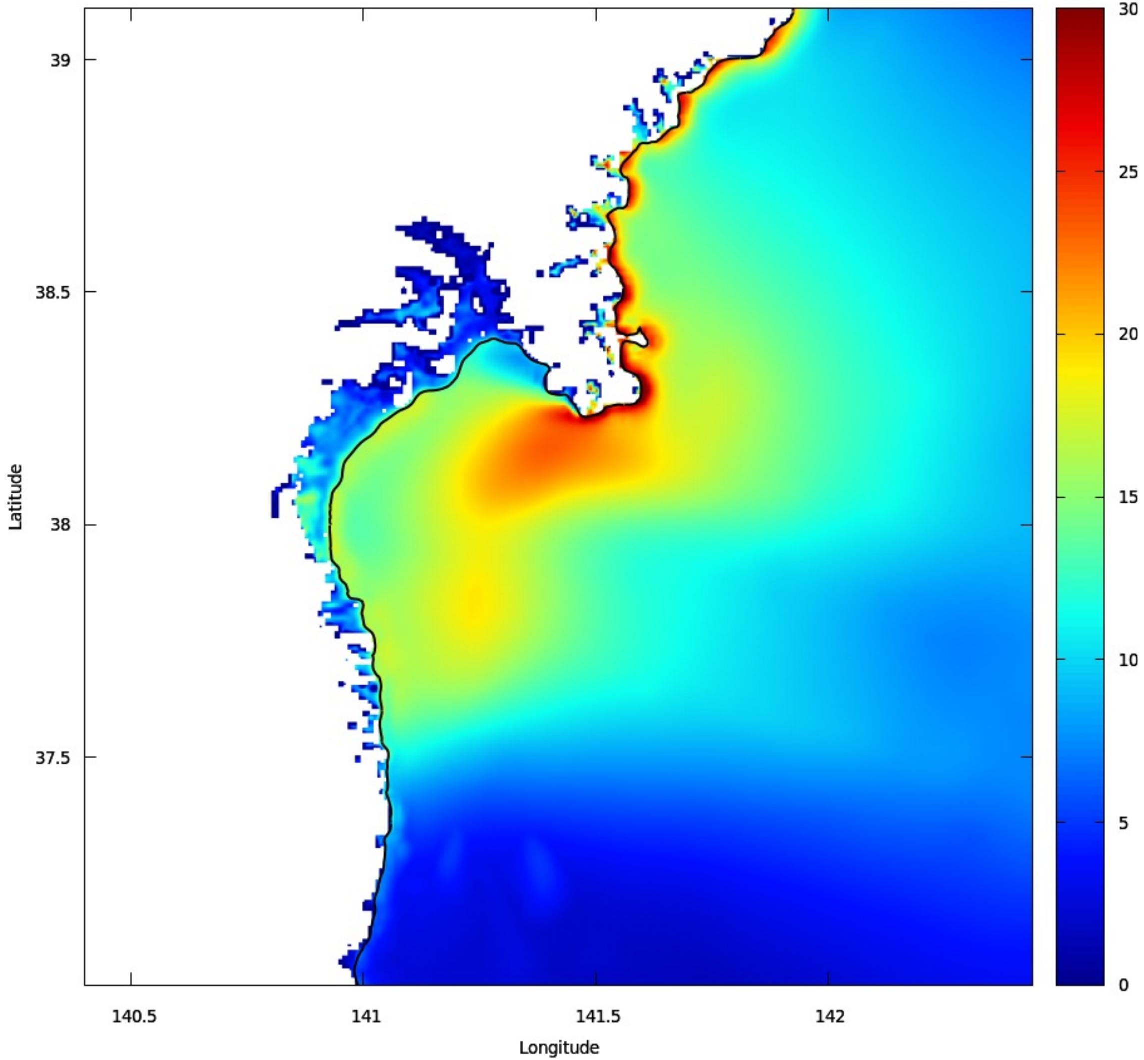
Tohoku tsunami: influence of dispersion

Serre–Green–Naghdi | One-layer non-hydrostatic | One-layer hydrostatic



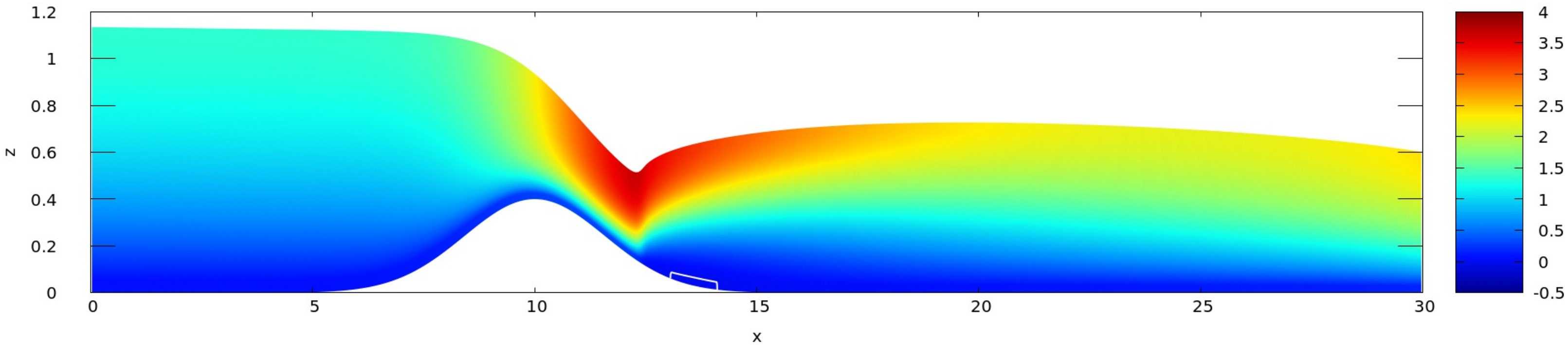
CPU runtime: 7h30 | CPU runtime: 4h15 | CPU runtime: 5h15

Computational speed $\approx$ 250 000 points $\times$ timesteps / sec

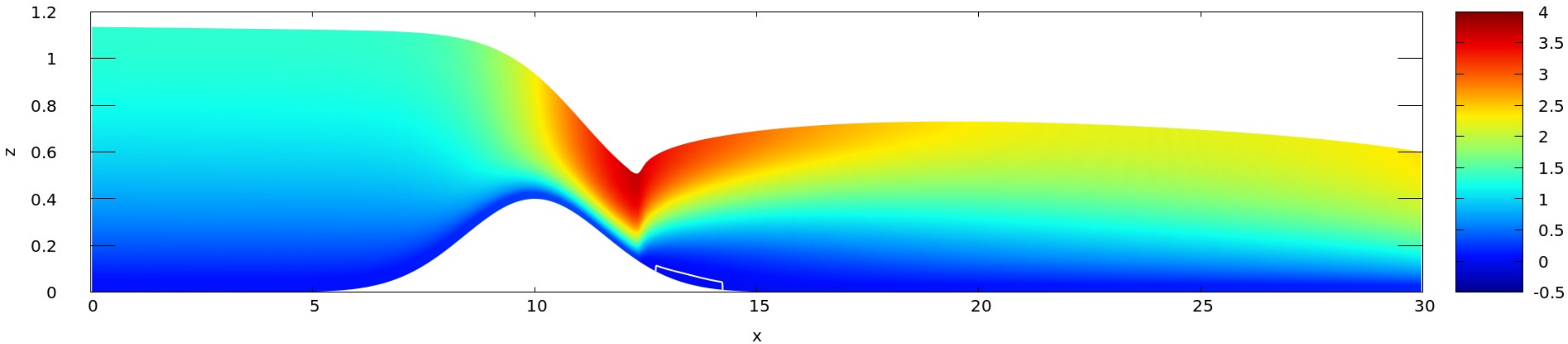


Sendai plain: 140×200 km

Horizontal velocity

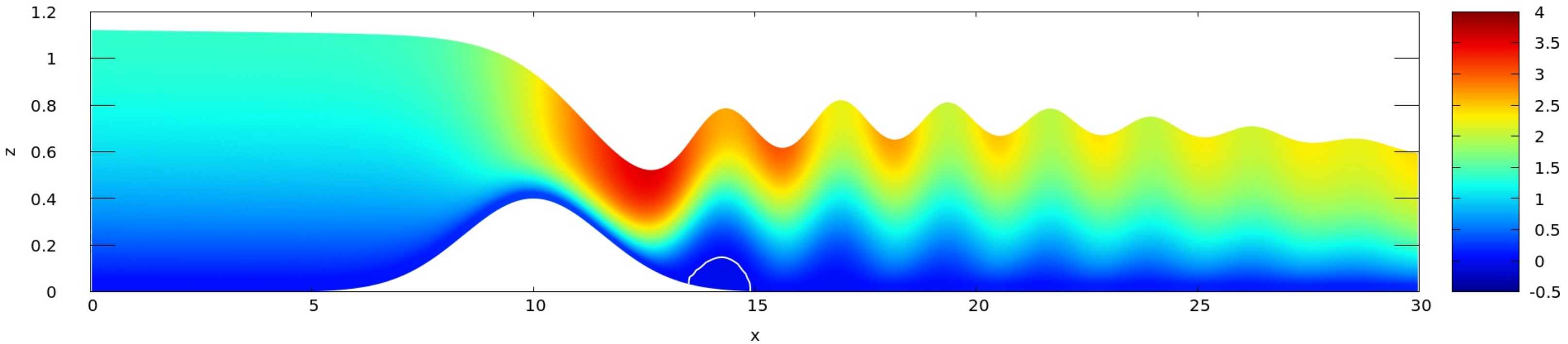


Multilayer hydrostatic

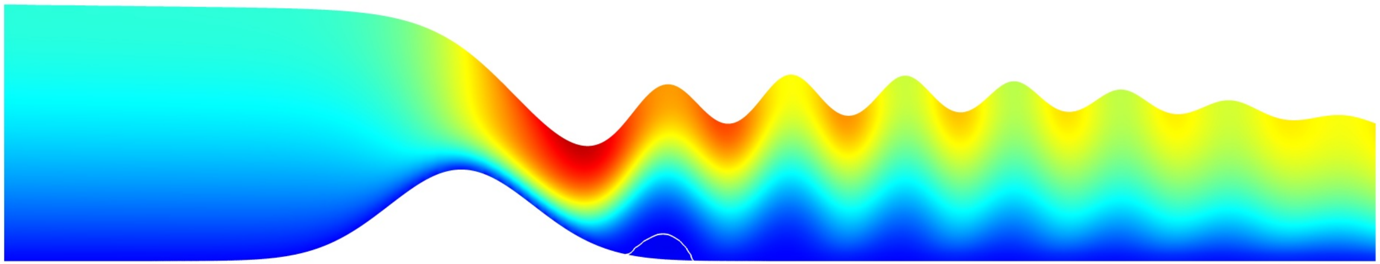


Riemann solver

Horizontal velocity

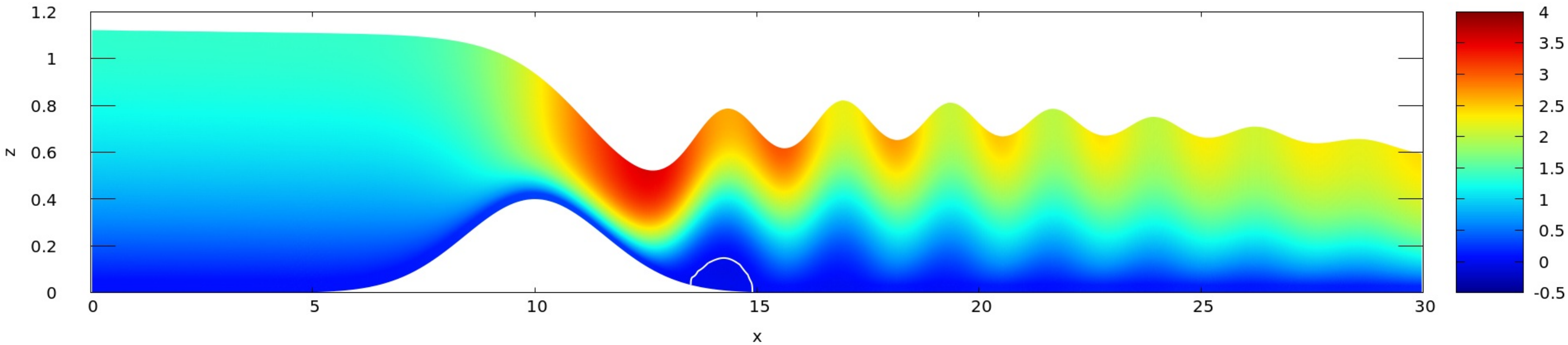


Multilayer non-hydrostatic

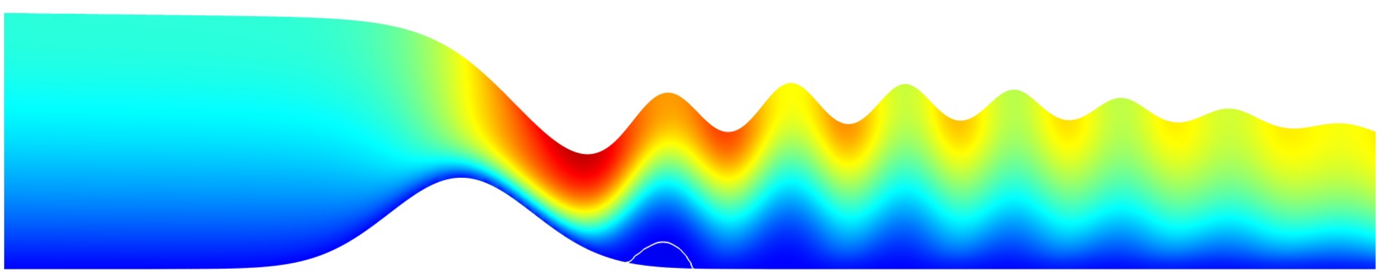


Navier–Stokes VOF

Horizontal velocity

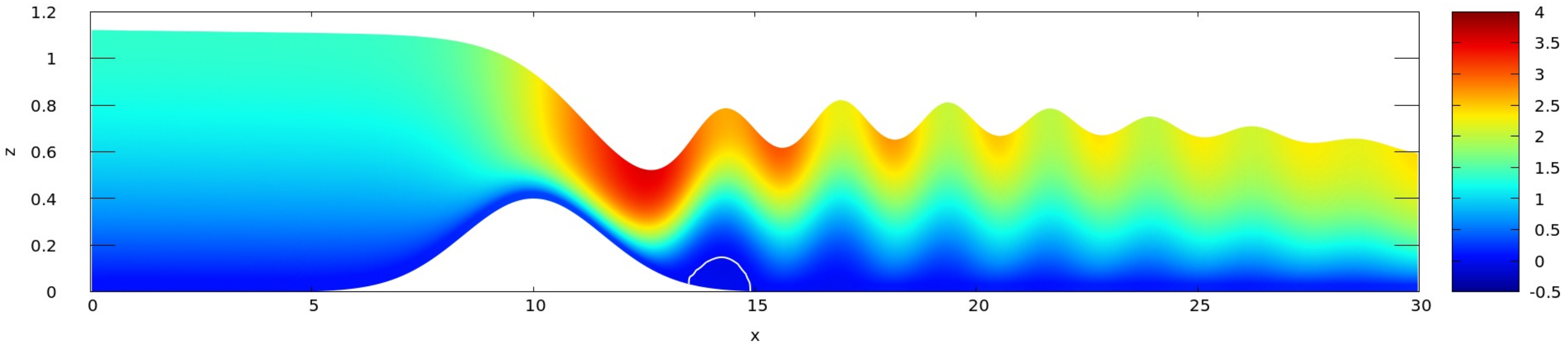


Multilayer non-hydrostatic (runtime: 3 min)

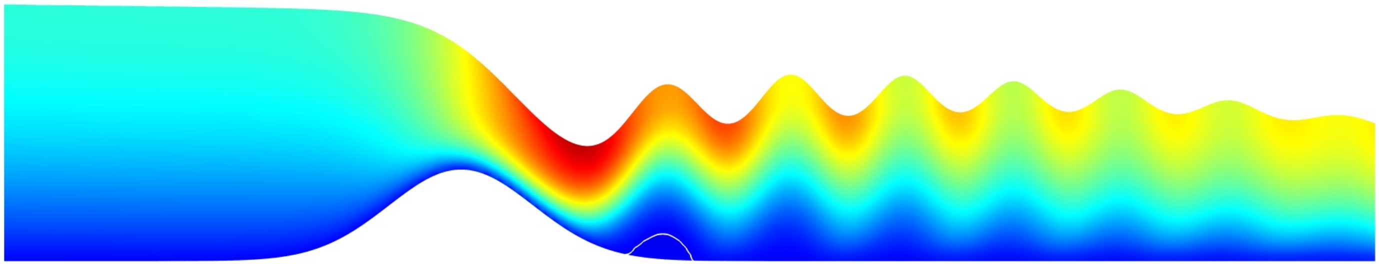


Navier–Stokes VOF

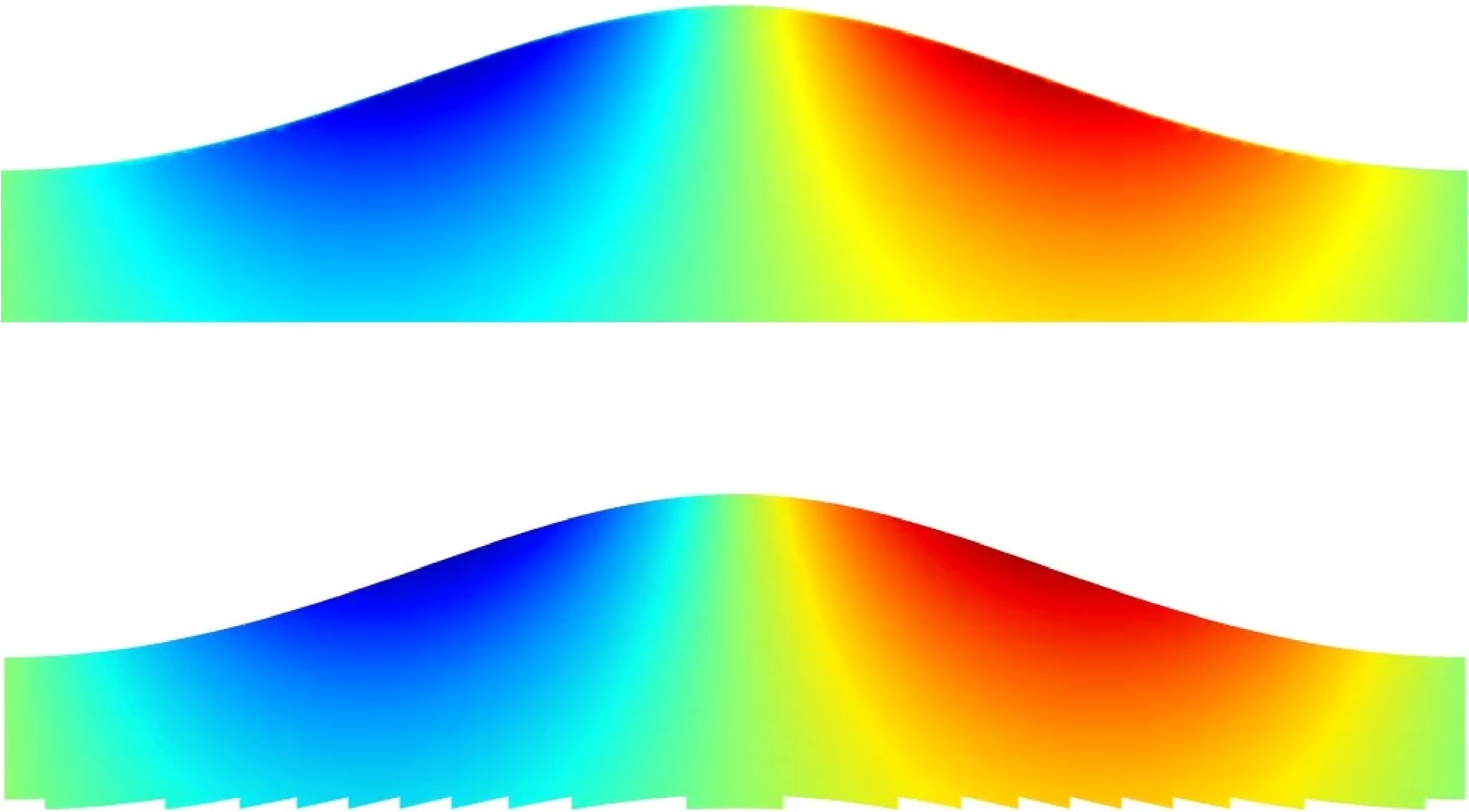
Horizontal velocity



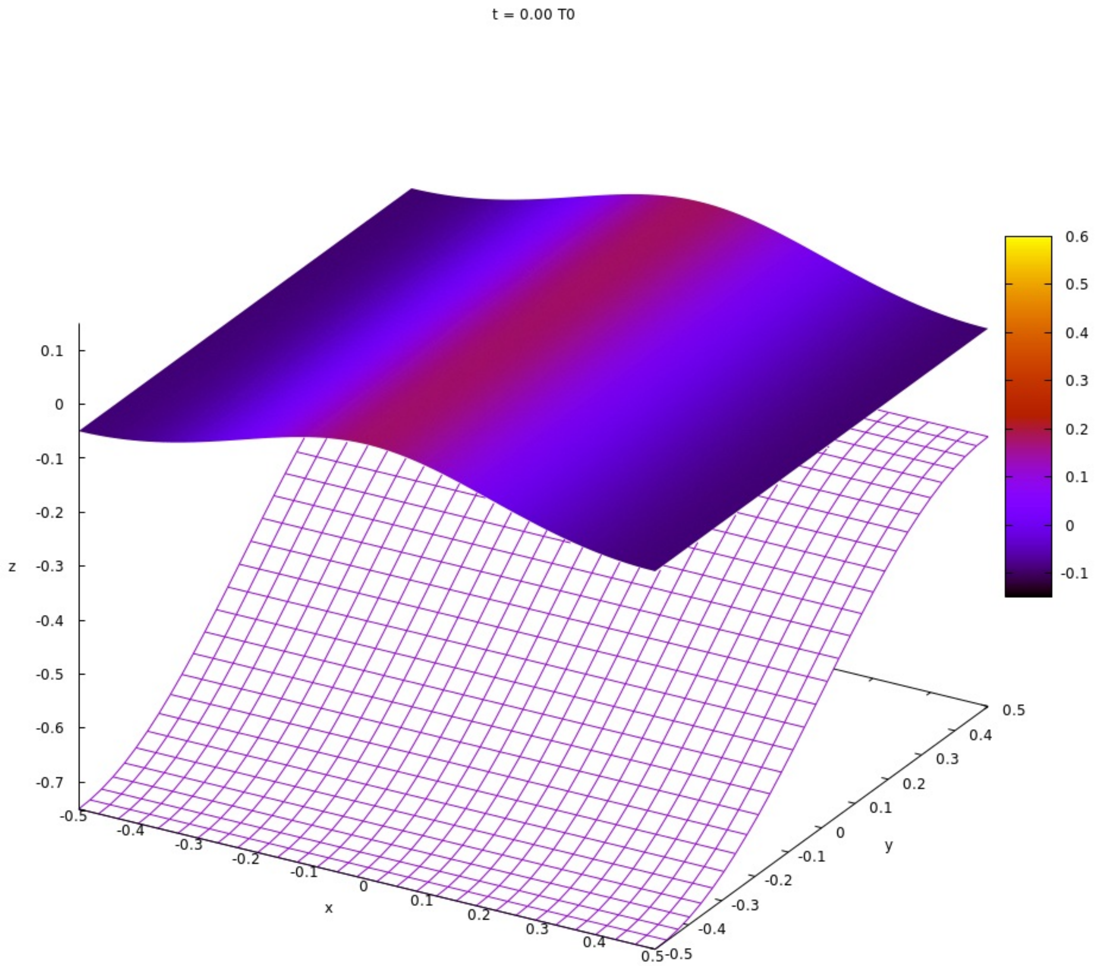
Multilayer non-hydrostatic (runtime: 3 min)



Navier–Stokes VOF (runtime: 1h15)



Vertical velocity: top: Navier–Stokes/VOF, bottom: multilayer



runtime: 13 minutes on 64 cores (MPI), $256^2 \times 80$

- A semi-discrete consistent representation of the incompressible Euler/Navier–Stokes equations with a free-surface.
- This (new) set of equations has a clear physical interpretation and makes a seamless link between the Euler, Saint-Venant and Boussinesq equations.
- The same model gives accurate and efficient solutions for the evolution of metre-scale to kilometer-scale waves.
- Work in progress:
  - Multimaterial flows: densities, rheologies, surface tension etc.
  - Coriolis forces / geostrophic balance
  - Applications to ocean modelling
- Preprint submitted to JCP on HAL and `basilisk.fr`

The choice of a Lagrangian vertical coordinates implies a kinematic condition i.e.

$$w = \partial_t z + \mathbf{u} \cdot \boldsymbol{\nabla} z$$

Note that this condition is physical only at the top and bottom boundaries. Using the vertical difference operator, the fact that $h_k = [z]_k$ and the continuity equation

$$\boldsymbol{\nabla} \cdot (h\,\mathbf{u})_k + [w - \mathbf{u} \cdot \boldsymbol{\nabla} z]_k = 0$$

we get

$$\begin{aligned} [w]_k &= \partial_t [z]_k + [\mathbf{u} \cdot \boldsymbol{\nabla} z]_k, \\ [w - \mathbf{u} \cdot \boldsymbol{\nabla} z]_k &= \partial_t h_k, \\ -\boldsymbol{\nabla} \cdot (h\,\mathbf{u})_k &= \partial_t h_k, \end{aligned}$$

which is the layer evolution equation.