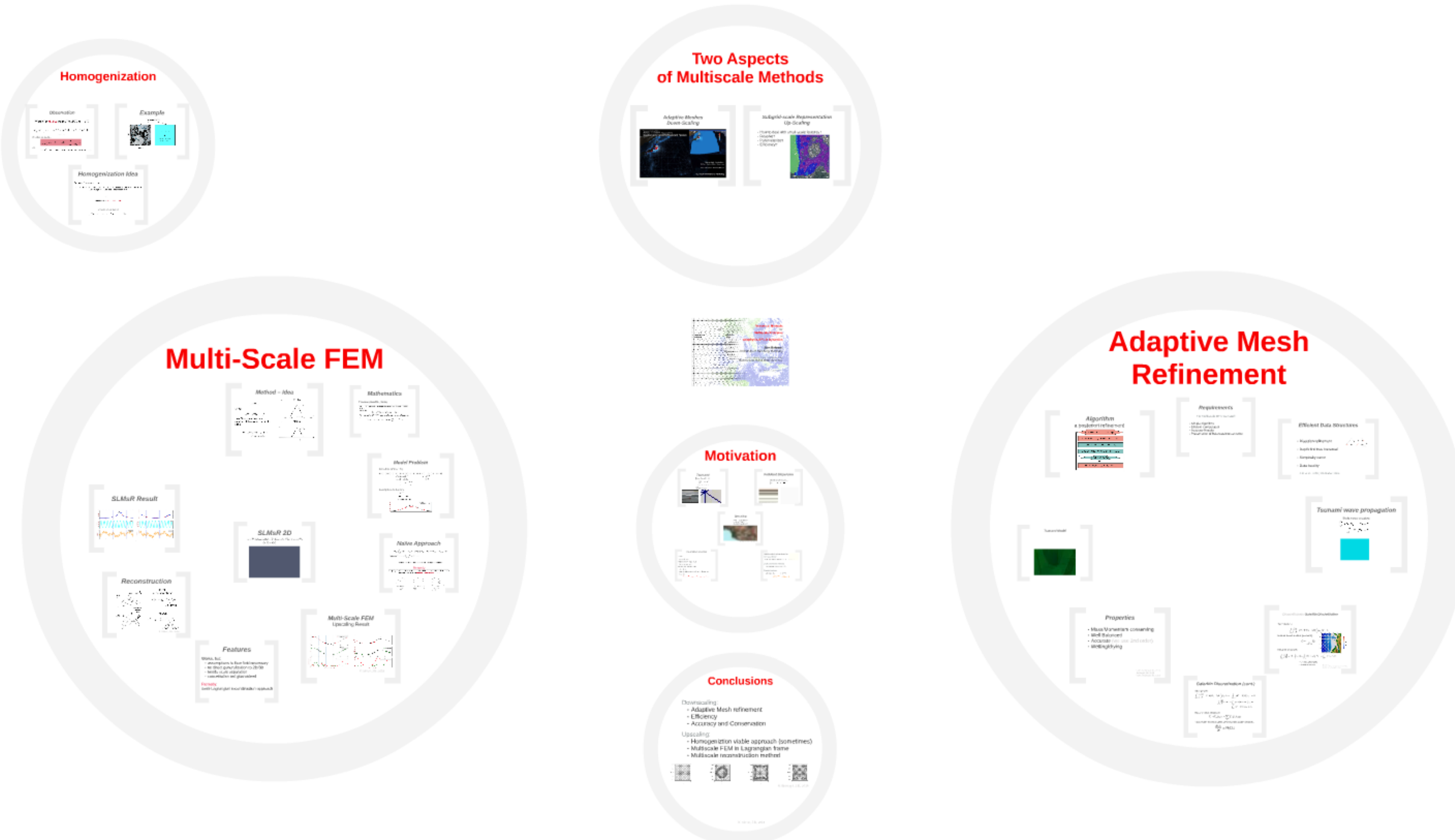


Homogenization
Example
Homogenization Idea
Two Aspects
of Multiscale Methods
Adaptive Meshes
Down-Scaling
Multi-Scale FEM
Method – Idea
Mathematics
Model Problem
SLMsR Result
SLMsR 2D
Naive Approach
Reconstruction
Features
Multi-Scale FEM
Upscaling Result
Motivation
Conclusions
Downscaling:
- Adaptive Mesh refinement
- Efficiency
- Accuracy and Conservation
Upscaling:
- Homogenization viable approach (sometimes)
- Multiscale FEM in Lagrangian frame
- Multiscale reconstruction method
Adaptive Mesh
Refinement
Algorithm
Requirements
Efficient Data Structures
Tsunami wave propagation
Properties
- Mass/Momentum conserving
- Well Balanced
- Accurate
- Wet/Dry
Galerkin Discretization

Numerical Methods for Multiscale Problems in Geophysical Fluid Dynamics

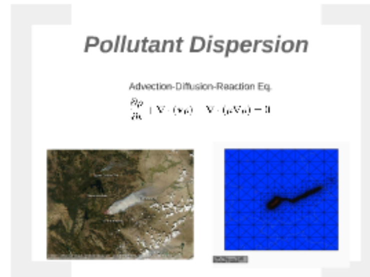
Jörn Behrens
University of Hamburg, Germany

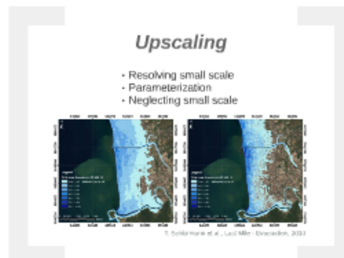
Acknowledging Major Contributions by:
Nicole Beisiegel, Konrad Simon, Stefan Vater

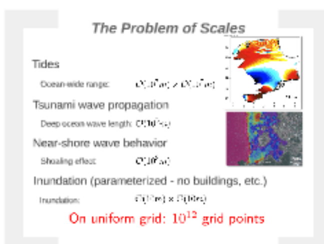
Support gratefully acknowledged:
Deutsche Forschungsgemeinschaft
German Research Foundation
Federal Ministry of Education and Research

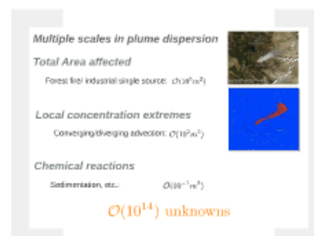
Motivation


Tsunami


Pollutant Dispersion
Advection-Diffusion-Reaction Eq.


Upscaling
• Resolving small scale
• Parameterization
• Neglecting small scale


The Problem of Scales
Tides
Tsunami wave propagation
Near-shore wave behavior
Inundation (parameterized - no buildings, etc.)
On uniform grid: 10^{12} grid points


Multiple scales in plume dispersion
Total Area affected
Local concentration extremes
Chemical reactions
$\mathcal{O}(10^{14})$ unknowns

Tsunami

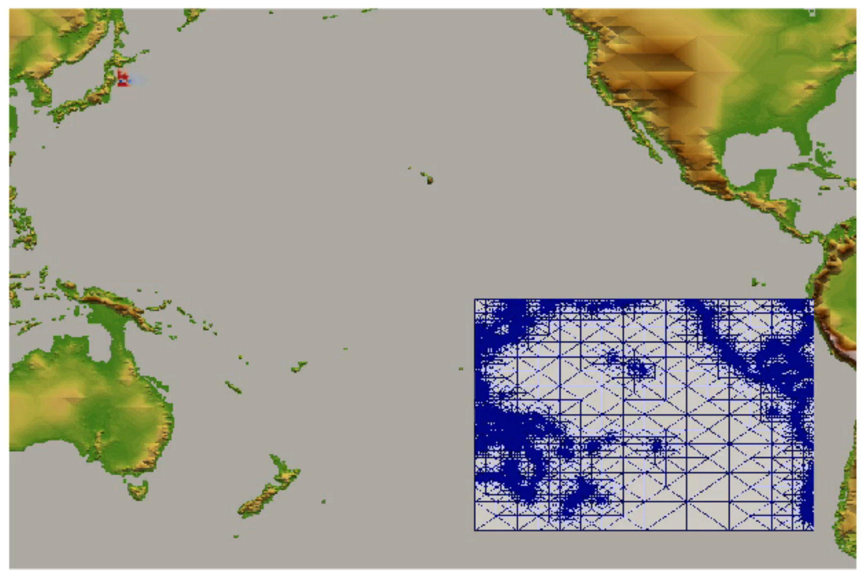
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + g\nabla\eta = R,$$

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (H\mathbf{v}) = 0.$$

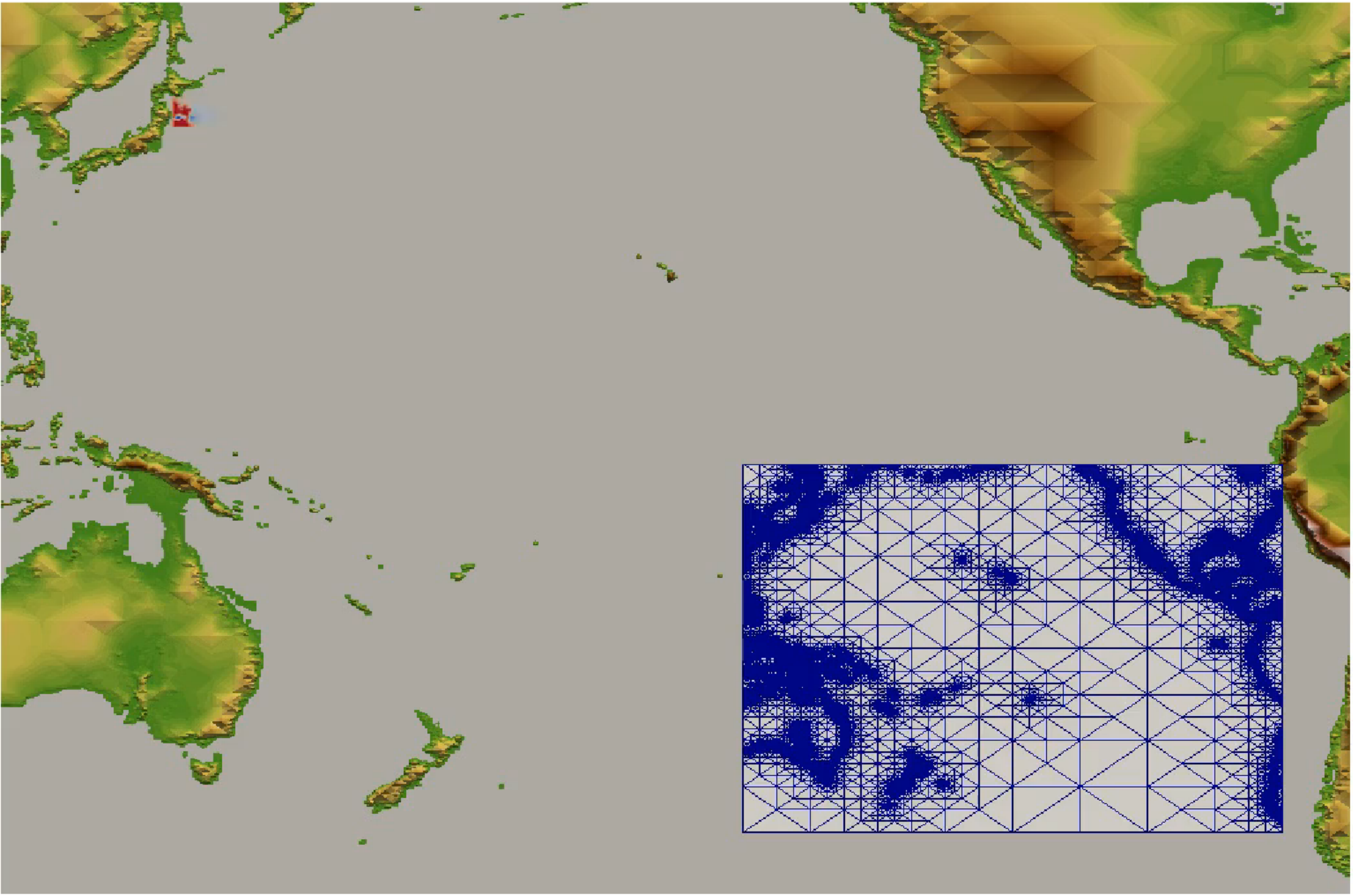
$$R = -f\mathbf{k} \times \mathbf{v} - rH^{-1}\mathbf{v}|\mathbf{v}| + H^{-1}\nabla(K_h H \nabla \mathbf{v})$$

Terms:

- Coriolis
- Bottom friction
- Viscosity (Smagorinsky approach)



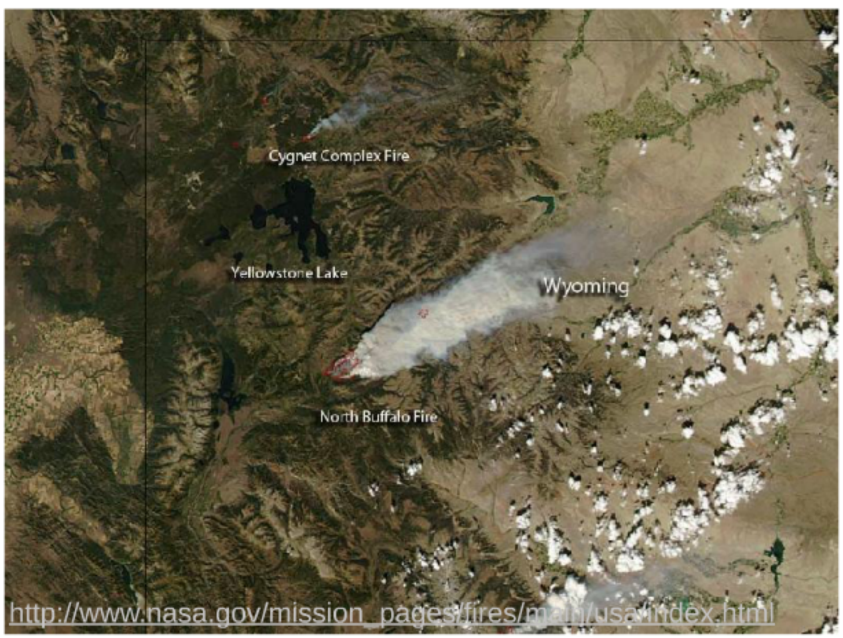
agorinsky approach)

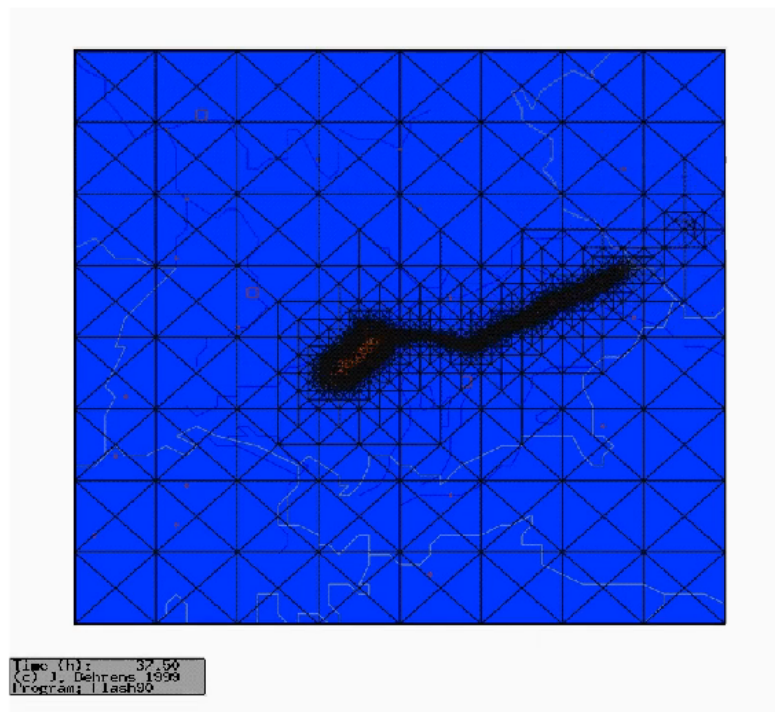


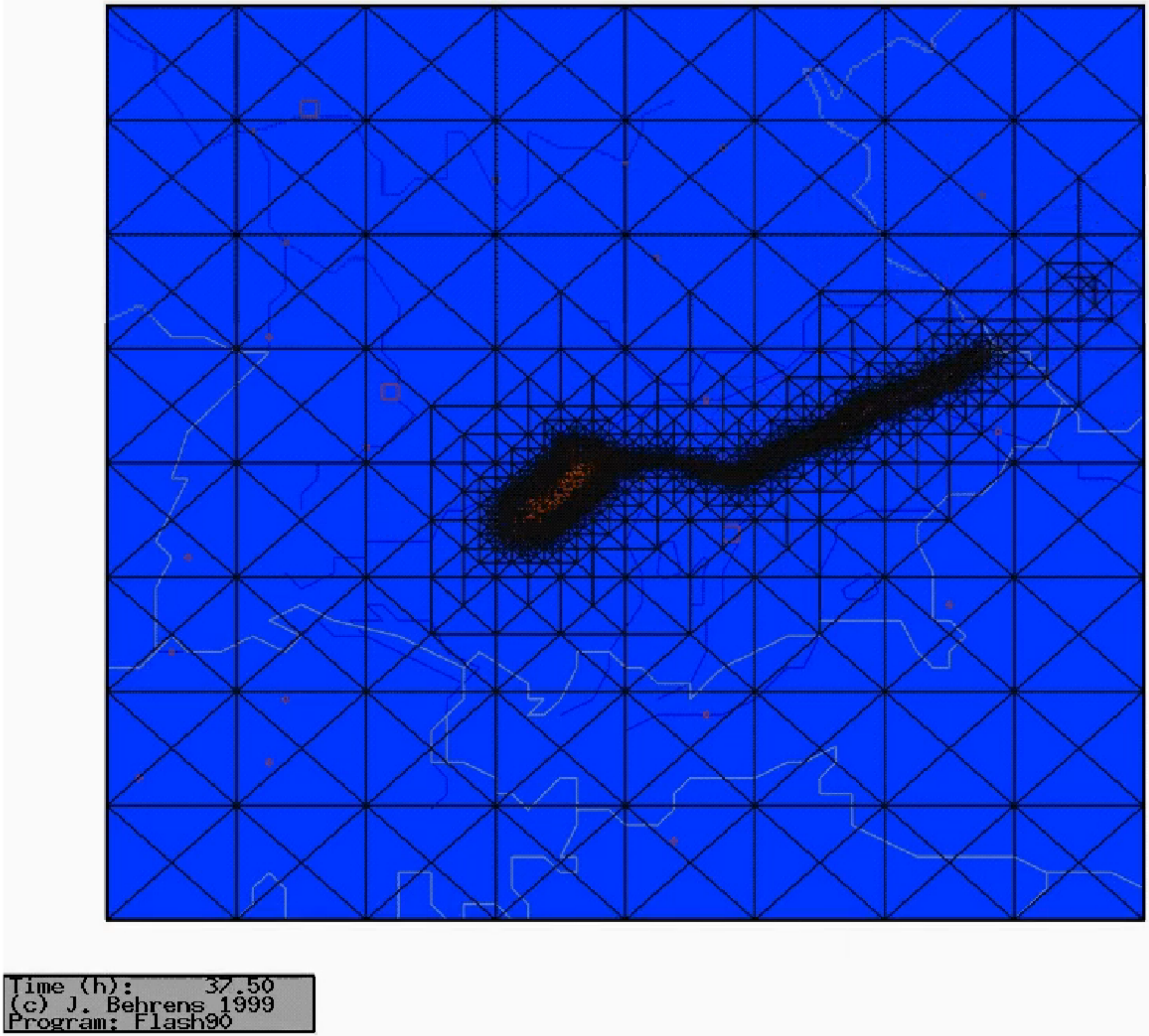
Pollutant Dispersion

Advection-Diffusion-Reaction Eq.

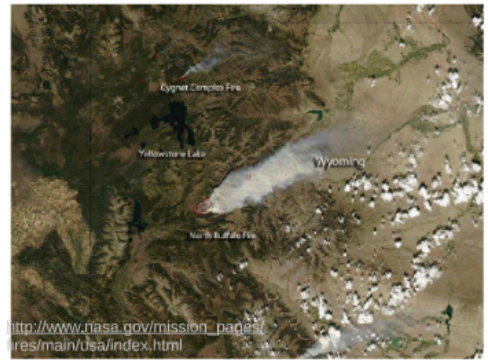
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) + \nabla \cdot (\mu \nabla \rho) = 0$$

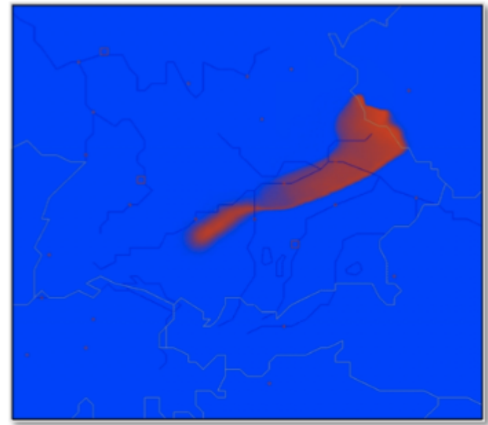





Time (h):   37.50
(c) J. Behrens 1999
Program: Flash90

Multiple scales in plume dispersion





Total Area affected

Forest fire/ industrial single source: $\mathcal{O}(10^5 m^2)$

Local concentration extremes

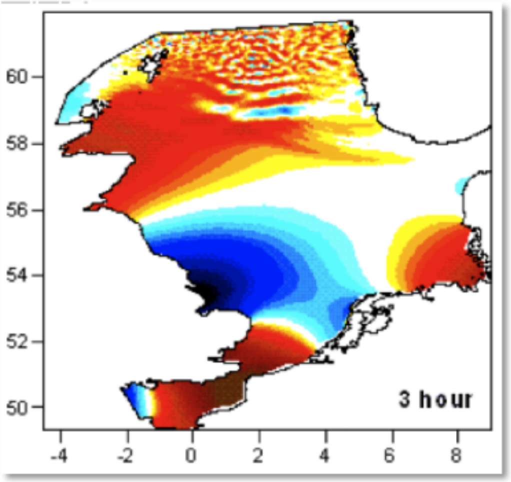
Converging/diverging advection: $\mathcal{O}(10^2 m^2)$

Chemical reactions

Sedimentation, etc.: $\mathcal{O}(10^{-1} m^2)$

$$\mathcal{O}(10^{14}) \text{ unknowns}$$

The Problem of Scales

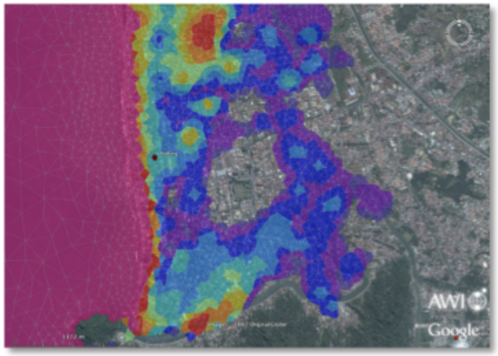


Tides

Ocean-wide range: $\mathcal{O}(10^7 m) \times \mathcal{O}(10^7 m)$

Tsunami wave propagation

Deep ocean wave length: $\mathcal{O}(10^5 m)$



Near-shore wave behavior

Shoaling effect: $\mathcal{O}(10^3 m)$

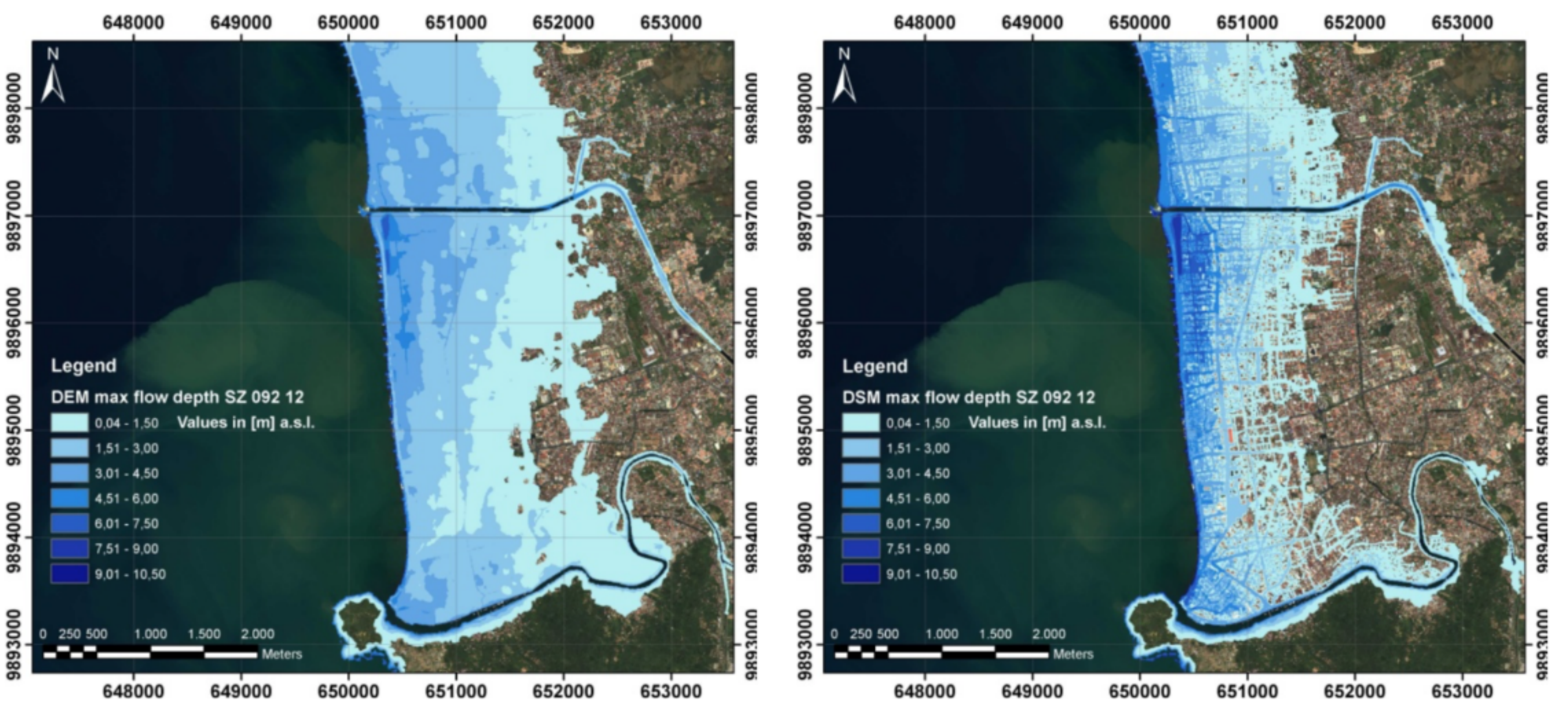
Inundation (parameterized - no buildings, etc.)

Inundation: $\mathcal{O}(10 m) \times \mathcal{O}(10 m)$

On uniform grid: 10^{12} grid points

Upscaling

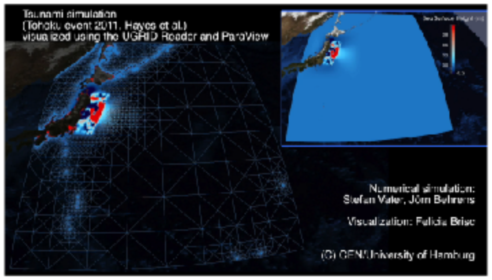
- Resolving small scale
- Parameterization
- Neglecting small scale



T. Schlurmann et al., Last Mile - Evacuation, 2010

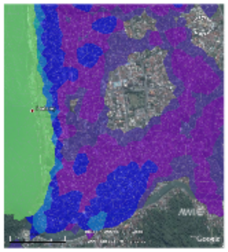
Two Aspects of Multiscale Methods

Adaptive Meshes
Down-Scaling

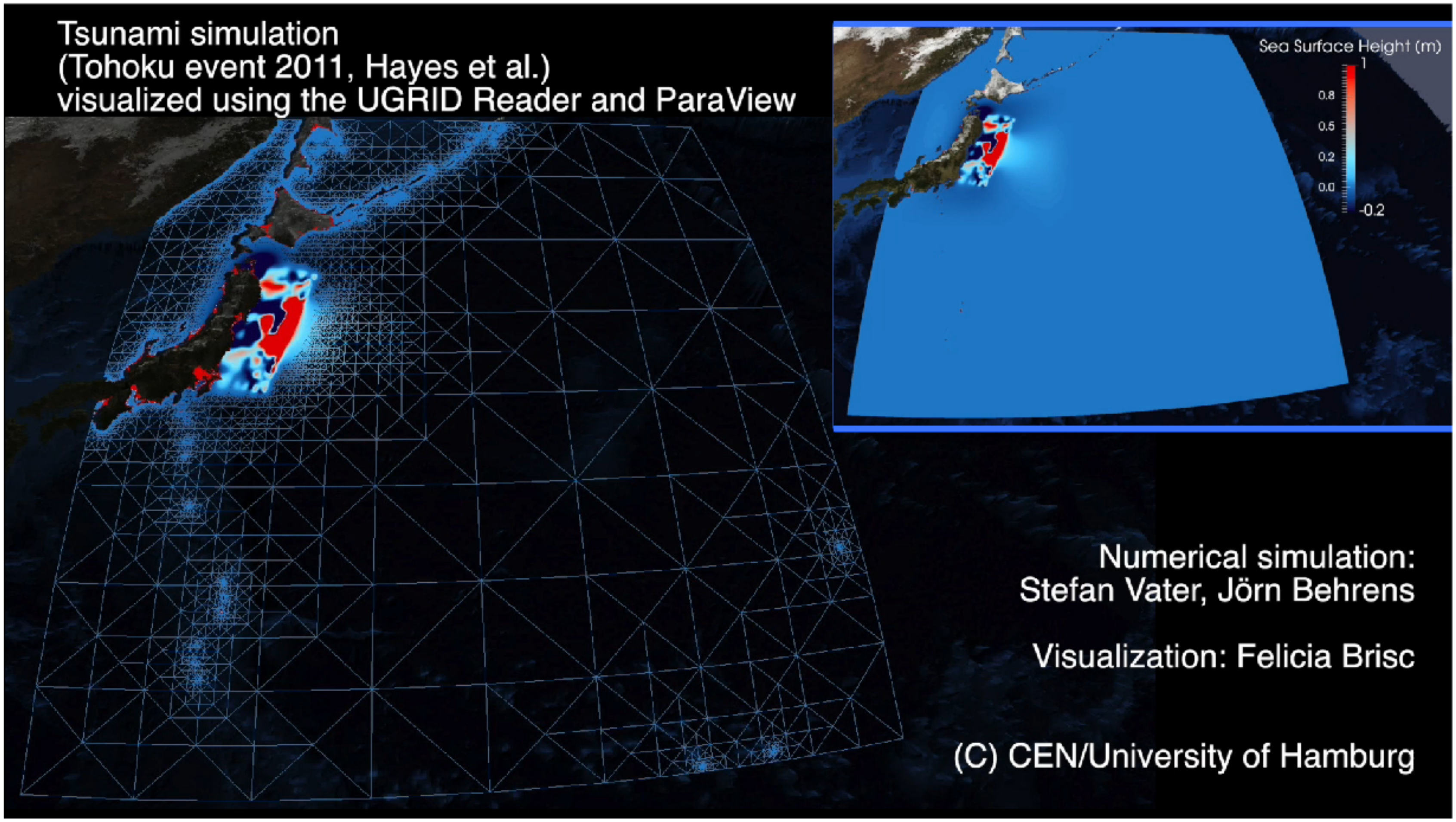


Subgrid-scale Representation
Up-Scaling

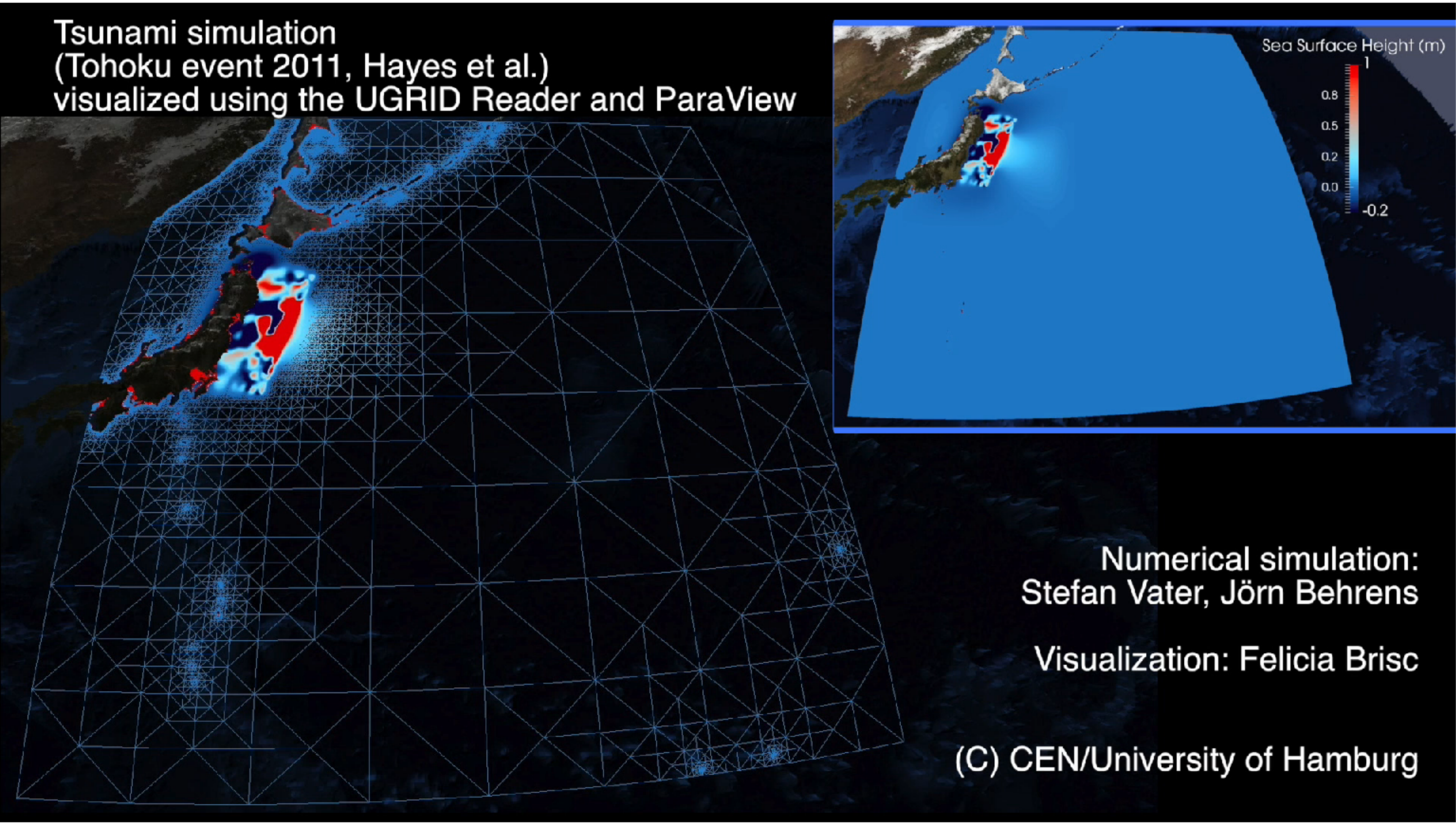
- How to deal with small-scale features?
- Resolve?
- Parameterize?
- Efficiency?



Adaptive Meshes
Down-Scaling

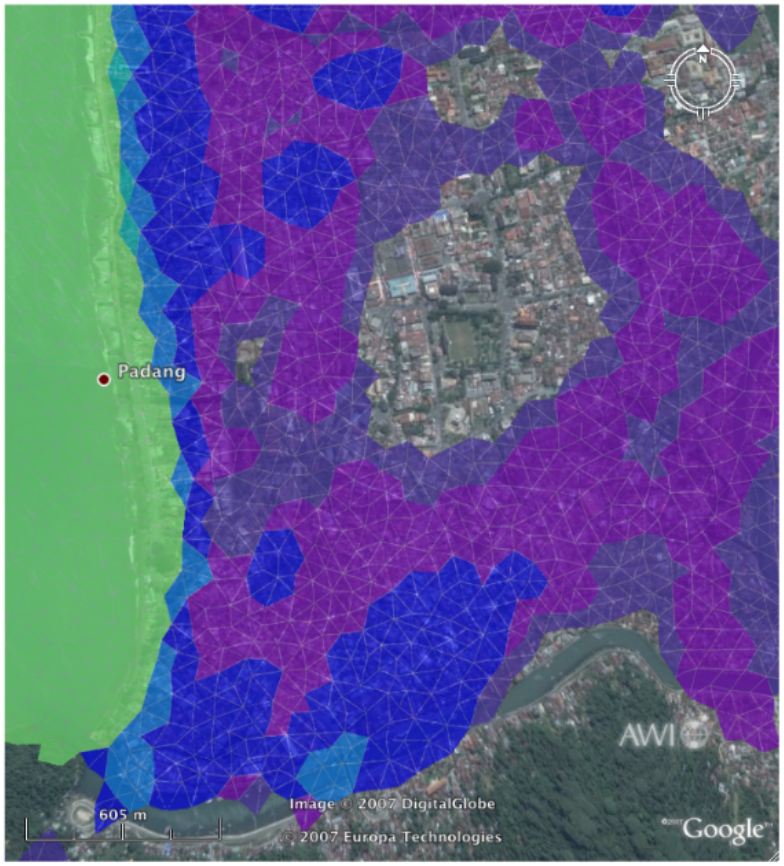


Down-Scaling



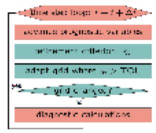
Subgrid-scale Representation Up-Scaling

- How to deal with small-scale features?
- Resolve?
- Parameterize?
- Efficiency?



Adaptive Mesh Refinement

Algorithm
a posteriori refinement



Requirements

For multiscale GFD we need:

- Simple Algorithm
- Efficient Computation
- Accurate Results
- Preservation of Balances/Conservation

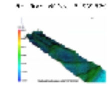
Efficient Data Structures

- Bisection refinement
- Depth first tree traversal
- Sierpinsky curve
- Data locality

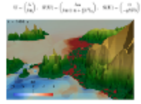
J.B. et al., 2005; J.B./Bader, 2009

Tsunami wave propagation

Shallow water equations

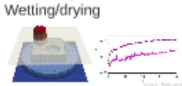


Tsunami Model



Properties

- Mass/Momentum conserving
- Well Balanced
- Accurate (we use 2nd order)
- Wetting/drying

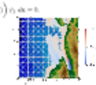


Discontinuous Galerkin Discretization

Test functions

Domain decomposition (elements)

Integration by parts



Galerkin Discretization (cont.)

Strong Form

Basis function expansion

Mass matrix inversion gives semi-discrete system of ODEs

$$\frac{\partial \mathbf{U}_h}{\partial t} = \mathbf{H}(\mathbf{U}_h)$$

Requirements

For multiscale GFD we need:

- Simple Algorithm
- Efficient Computation
- Accurate Results
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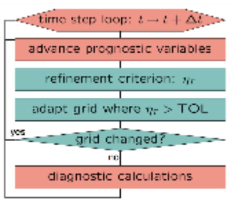
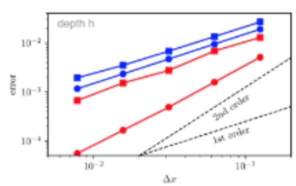
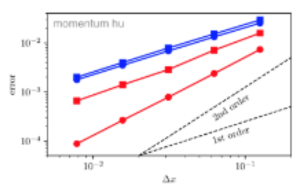
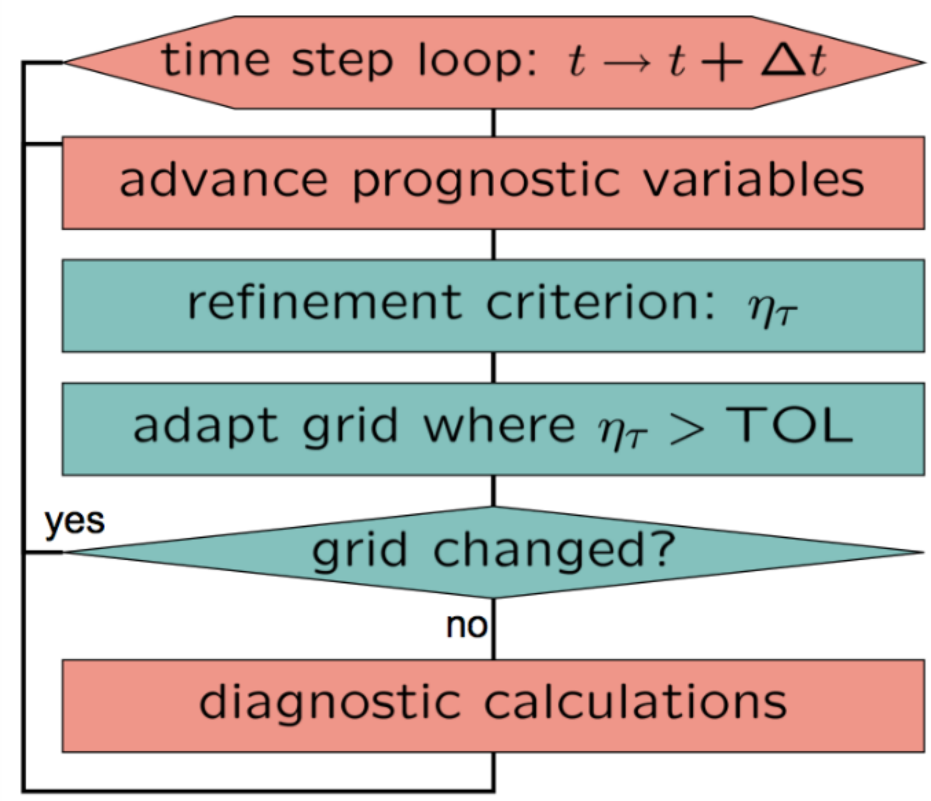




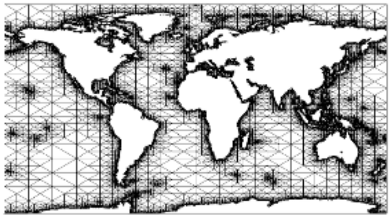


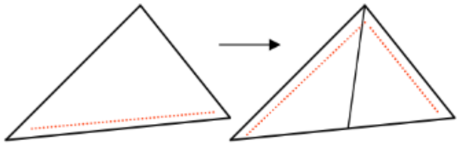
Algorithm

a posteriori refinement

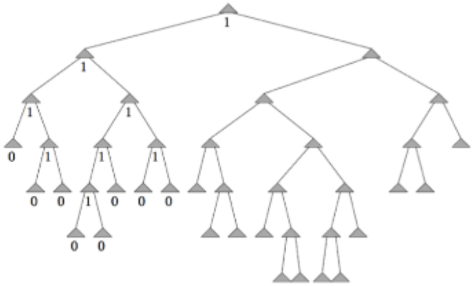


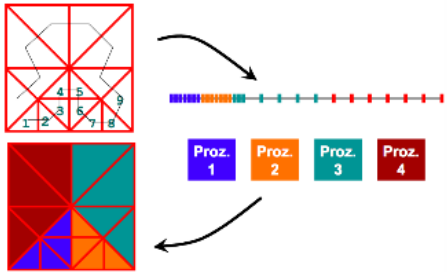
Efficient Data Structures





Rivara (1984)





- Bisection refinement
- Depth first tree traversal
- Sierpinsky curve
- Data locality

J.B. et al., 2005; J.B./Bader, 2009

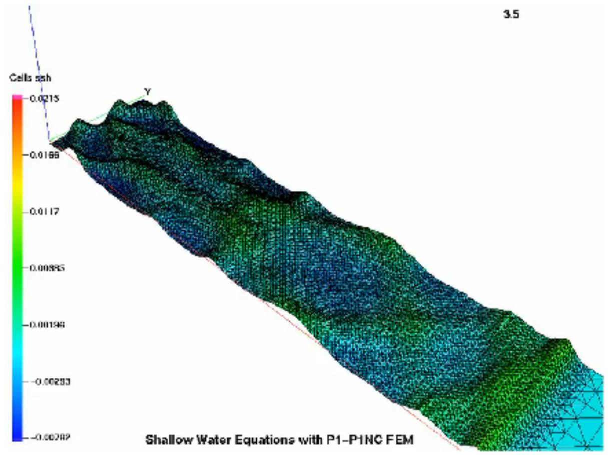
Tsunami wave propagation

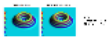
Shallow water equations

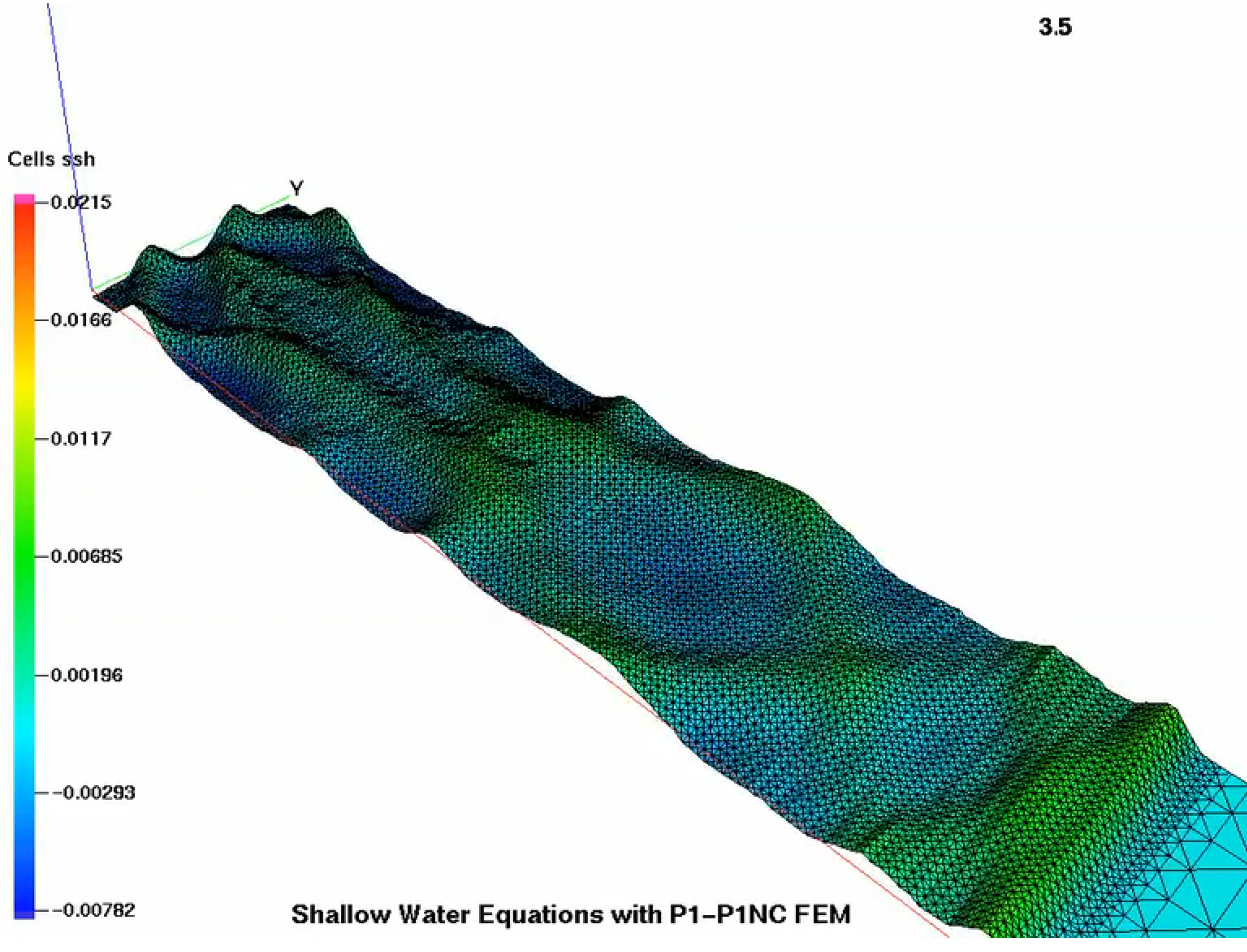
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + g\nabla\eta = R,$$

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (H\mathbf{v}) = 0.$$

$$R = -f\mathbf{k} \times \mathbf{v} - rH^{-1}\mathbf{v}|\mathbf{v}| + H^{-1}\nabla(K_h H \nabla \mathbf{v})$$

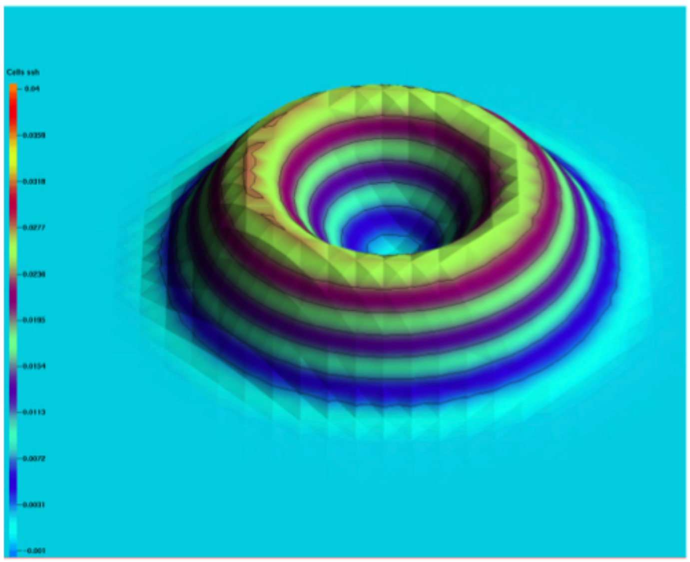




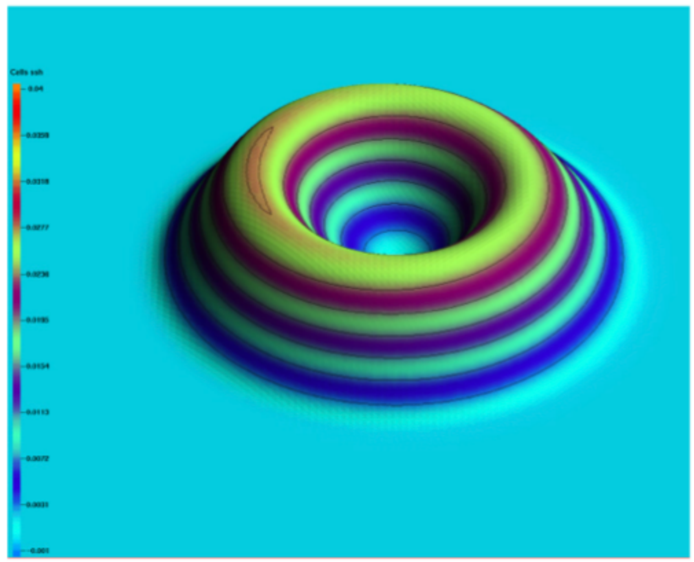


3.5
Cells ssh
Y
0.0215
0.0166
0.0117
0.00685
0.00196
−0.00293
−0.00782
Shallow Water Equations with P1−P1NC FEM

20480 cells uniform



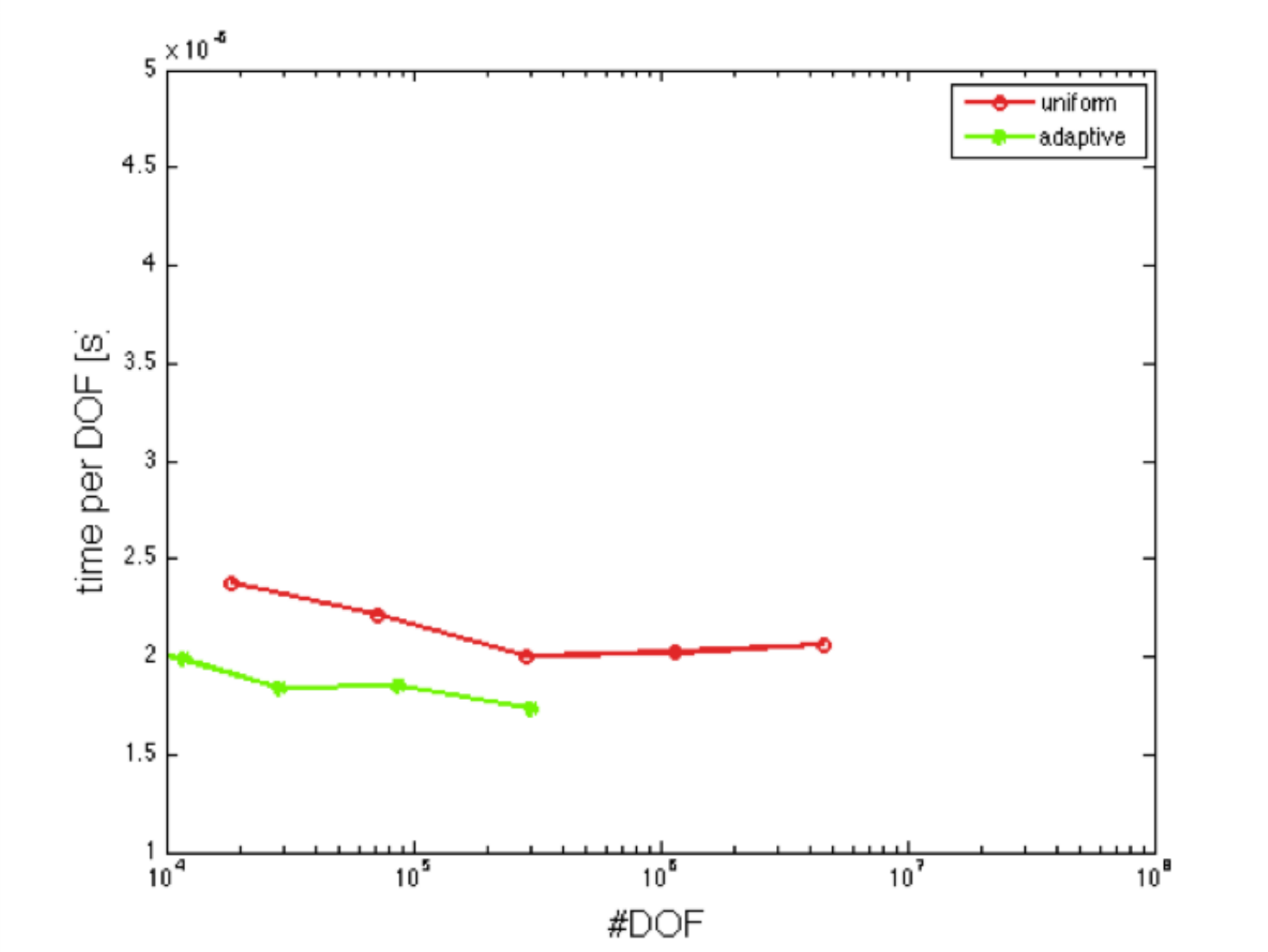
10576..24662 cells uniform



FEM:

Adaptive: 15.8 s

Uniform: 173 s



$\times 10^{-5}$
5
4.5
4
3.5
3
2.5
2
1.5
1
time per DOF [s]
10^4
10^5
10^6
10^7
10^8
#DOF
uniform
adaptive

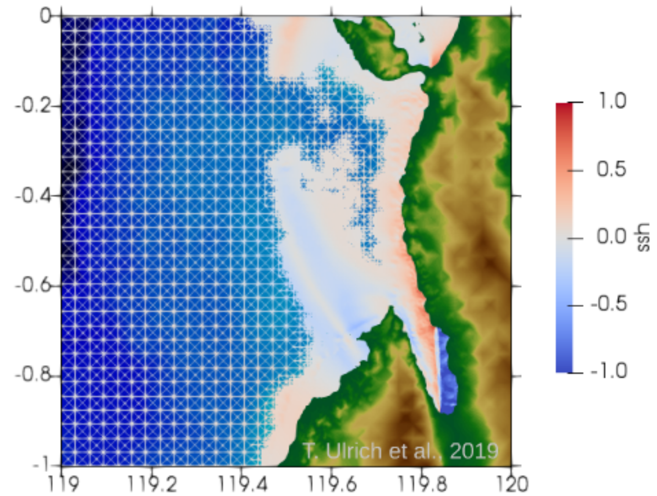
Discontinuous ***Galerkin Discretization***

Test functions

$$\int_{\Omega}\left(\frac{\partial \mathbf{U}}{\partial t}+\nabla\cdot\mathbf{F}(\mathbf{U})-S(\mathbf{U})\right)\psi_j\ d\mathbf{x}=0.$$

Domain decomposition (elements)

$$\Omega=\bigcup_{e=1:M}\Omega_e$$



Integration by parts

$$\int_{\Omega_e}\left(\frac{\partial \mathbf{U}}{\partial t}-S(\mathbf{U})\right)\psi_j\ d\mathbf{x}-\int_{\Omega_e}\mathbf{F}(\mathbf{U})\cdot\nabla\psi_j\ d\mathbf{x}=-\oint_{\partial\Omega_e}\mathbf{F}^*\psi_j\cdot\mathbf{n}\ dS.$$

- $\mathbf{F}^*$ a numerical flux function
- $\mathbf{n}$ outward normal vector

References:
Hesthaven (1998), Hesthaven&Warburton (2006),
Giraldo et al. (2002), Giraldo (2006).

Galerkin Discretization (cont.)

Strong Form

$$\int_{\Omega_e} \left(\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) - S(\mathbf{U}) \right) \psi_j \, d\mathbf{x} = - \oint_{\partial \Omega_e} \left(\mathbf{F}^* - \mathbf{F}(\mathbf{U}) \right) \psi_j \cdot \mathbf{n} \, dS.$$

$$\begin{aligned} \int_{\Omega_e} \frac{\partial \mathbf{U}}{\partial t} \psi_j \, d\mathbf{x} = & - \int_{\Omega_e} \left(\nabla \cdot \mathbf{F}(\mathbf{U}) - S(\mathbf{U}) \right) \psi_j \, d\mathbf{x} \\ & - \oint_{\partial \Omega_e} \left(\mathbf{F}^* - \mathbf{F}(\mathbf{U}) \right) \psi_j \cdot \mathbf{n} \, dS. \end{aligned}$$

Basis function expansion

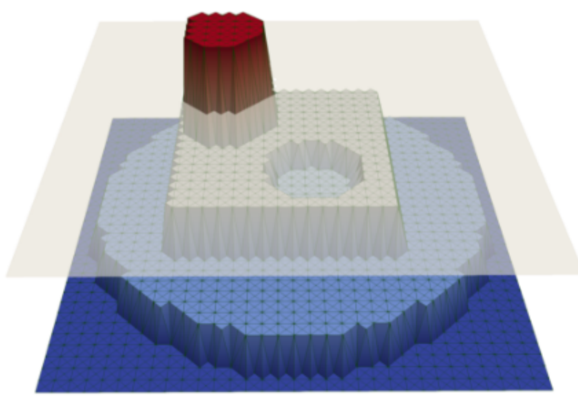
$$\mathbf{U} \approx \mathbf{U}_h(\mathbf{x}, t) = \sum_k \mathbf{U}_k(t) \psi_k(\mathbf{x})$$

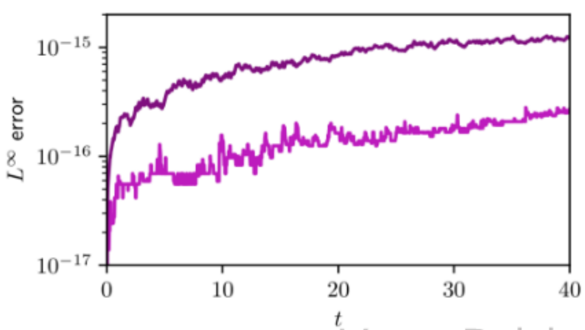
Mass matrix inversion gives semi-discrete system of ODEs

$$\frac{\partial \mathbf{U}_h}{\partial t} = \mathbf{H}(\mathbf{U}_h)$$

Properties

- Mass/Momentum conserving
- Well Balanced
- Accurate (we use 2nd order)
- Wetting/drying





Vater, Beisiegel, JB, 2015;
Beisiegel, JB, 2015;
Vater, Beisiegel, JB, 2019.

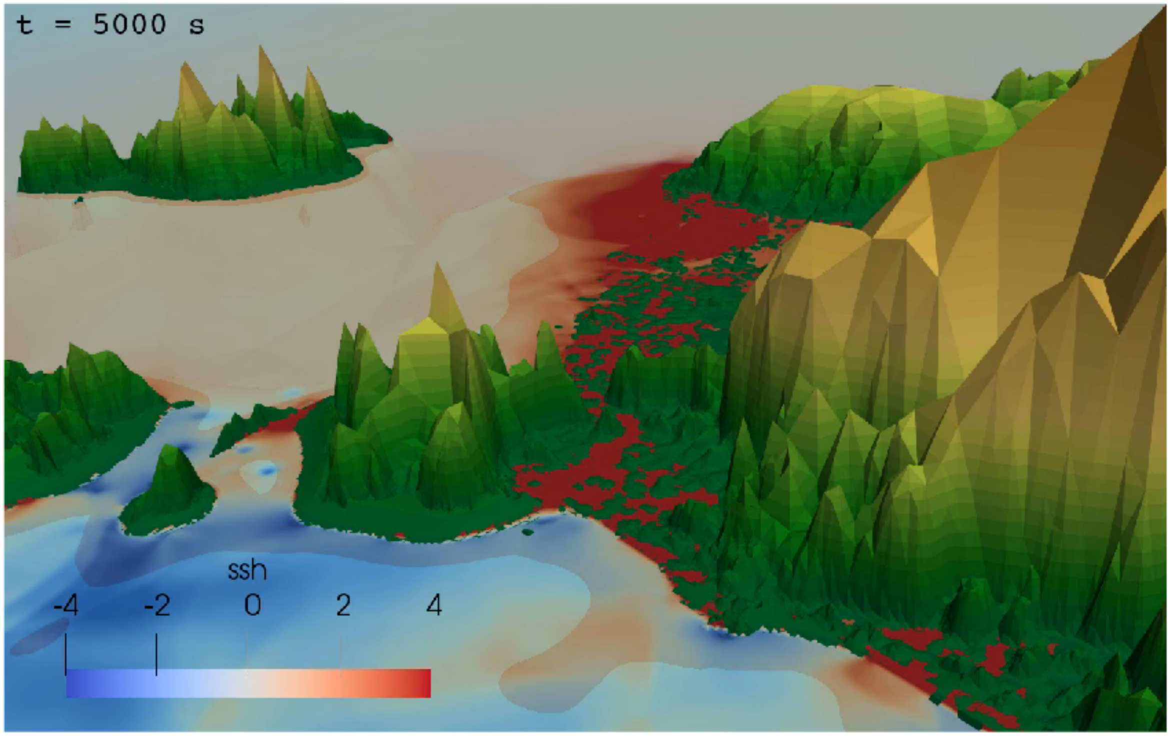
Tsunami Model

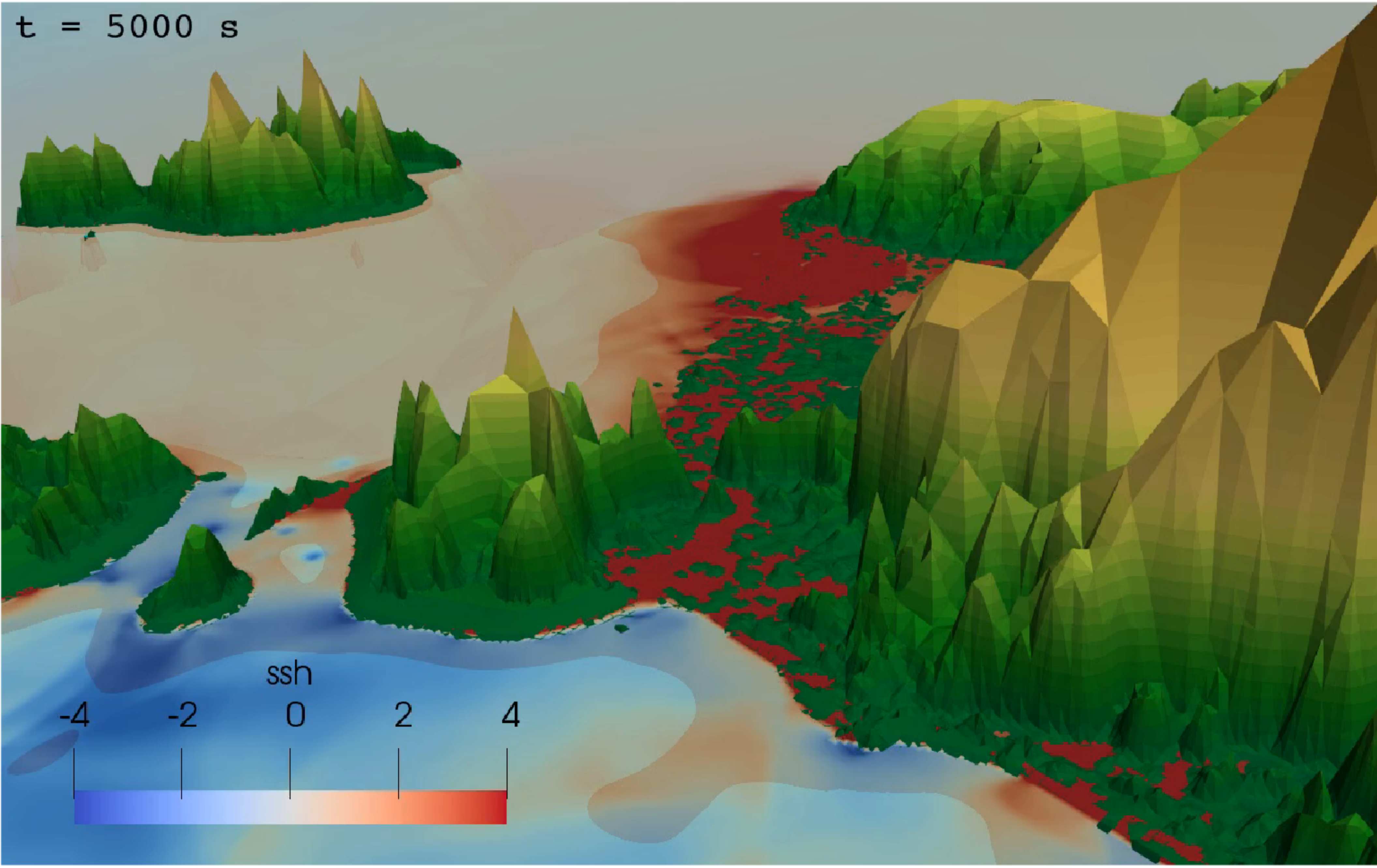
Solve

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U})$$

with

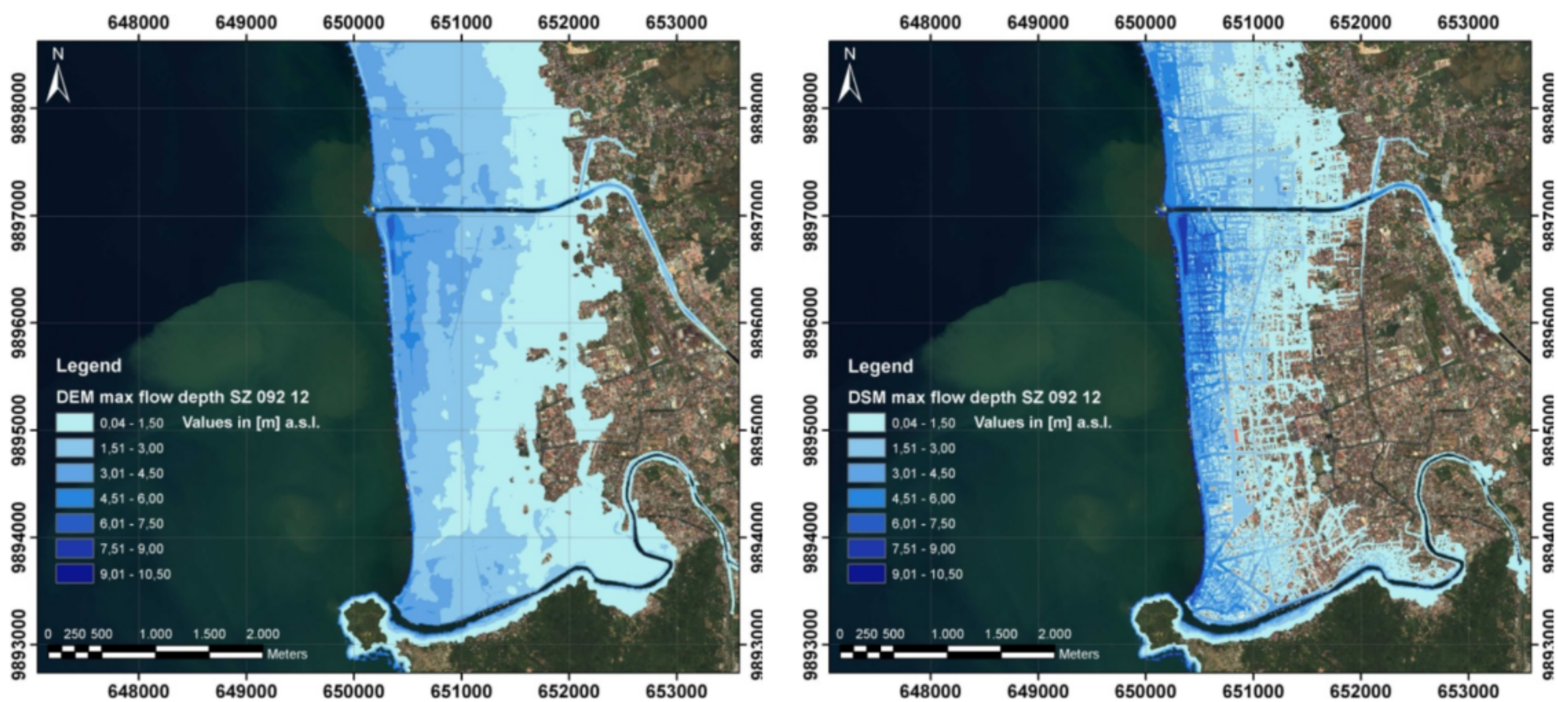
$$\mathbf{U} = \begin{pmatrix} h \\ h\mathbf{u} \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} h\mathbf{u} \\ h\mathbf{u} \otimes \mathbf{u} + \frac{g}{2}h^2\mathbb{I}_2 \end{pmatrix}, \quad \mathbf{S}(\mathbf{U}) = \begin{pmatrix} 0 \\ -gh\nabla b \end{pmatrix}$$




t = 5000 s
ssh
-4
-2
0
2
4

Upscaling

- Resolving small scale
- Parameterization
- Neglecting small scale



T. Schlurmann et al., Last Mile - Evacuation, 2010

Homogenization

Observation

What is the effective model of this PDE as $\varepsilon \to 0$?

$$\left(a(\frac{x}{\varepsilon})u^\varepsilon_x\right)_x = f\,, \quad x \in I = [a,b]\,, 0 < q \le a \in L^\infty([0,1])$$

The effective model is

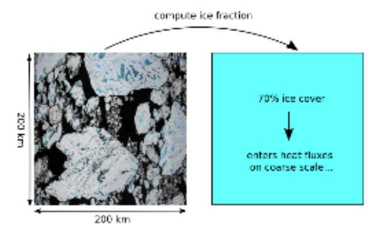
$$\boxed{\frac{1}{m_H(a)}u^*_{xx} = f\,, \quad m_H(a) = \int_0^1 \frac{1}{a(y)}\,\mathrm{d}y}$$

not

$$m_A(a)u^*_{xx} = f\,, \quad m_A(a) = \int_0^1 a(y)\,\mathrm{d}y$$

In general $m_A(a) \ge \frac{1}{m_H(a)}$!

Example



Homogenization Idea

We have $F_\varepsilon(u_\varepsilon) = 0$ so maybe

Find F^* and u^* so that $u_\varepsilon \to u^*$ and $F_\varepsilon \to F^*$ (in some sense) in the limit of large range of scales, scale separation with

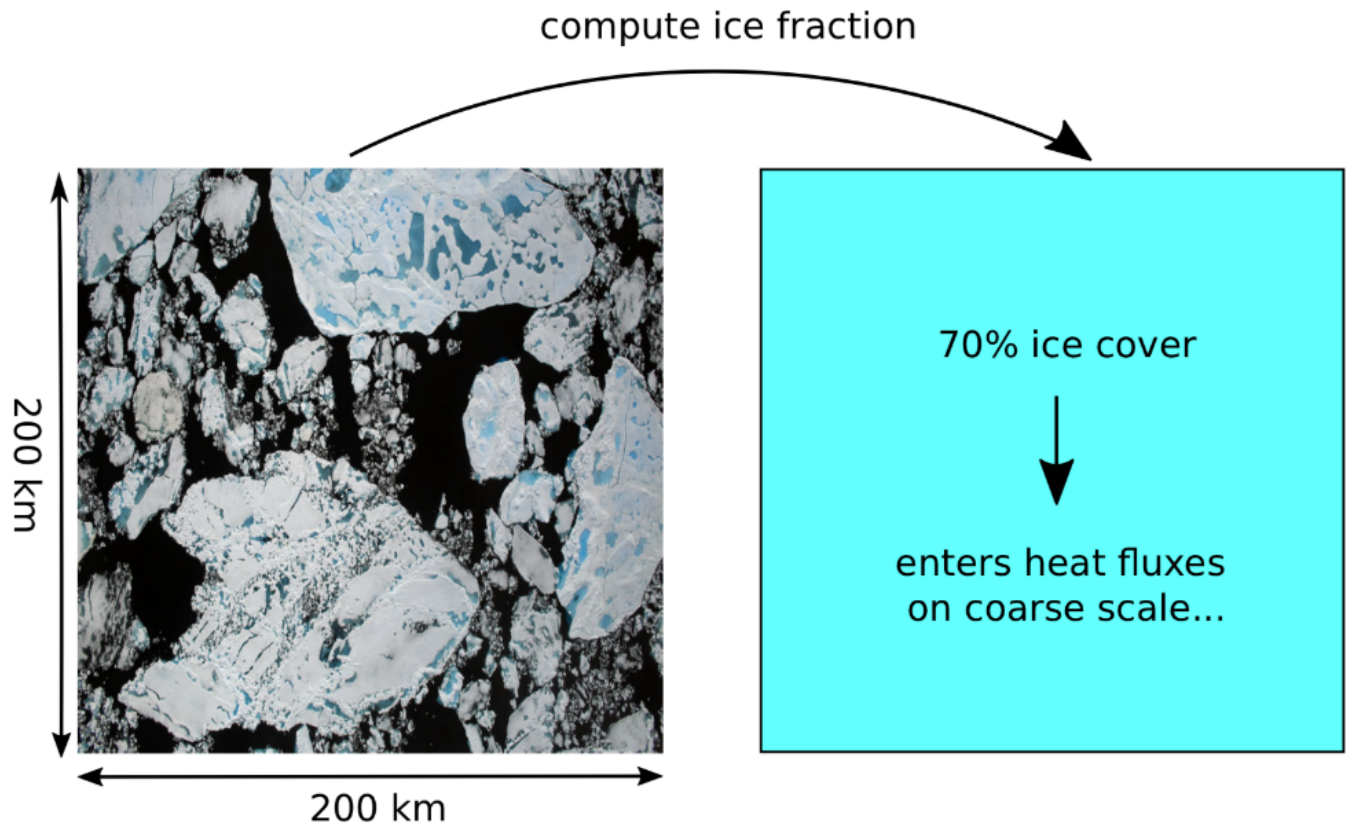
$$F^*(u^*) = 0\,.$$

This is called effective model.

Assume as our model

$$\left(a(\frac{x}{\varepsilon})u^\varepsilon_x\right)_x = f\,, \quad x \in I = [a,b]\,, 0 < q \le a \in L^\infty([0,1])$$

Example


compute ice fraction
200 km
200 km
70% ice cover
enters heat fluxes
on coarse scale...

Homogenization Idea

We have $F_\varepsilon(u_\varepsilon) = 0$ so maybe

Find F^* and u^* so that $u_\varepsilon \to u^*$ and $F_\varepsilon \to F^*$ (in some sense) in the limit of large range of scales, scale separation with

$$F^*(u^*) = 0 \; .$$

This is called effective model.

Assume as our model

$$\left(a(\frac{x}{\varepsilon})u_x^\varepsilon\right)_x = f \; , \quad x \in I = [a,b] \; , 0 < q \leq a \in L^\infty([0,1])$$

Observation

What is the effective model of this PDE as $\varepsilon \to 0$?

$$\left(a(\frac{x}{\varepsilon})u_x^\varepsilon\right)_x = f\,, \quad x \in I = [a,b]\,, 0 < q \le a \in L^\infty([0,1])$$

The effective model is

$$\boxed{\frac{1}{m_H(a)} u_{xx}^* = f\,, \quad m_H(a) = \int_0^1 \frac{1}{a(y)}\,\mathrm{d}y}$$

not

$$m_{\mathsf{A}}(a) u_{xx}^* = f\,, \quad m_{\mathsf{A}}(a) = \int_0^1 a(y)\,\mathrm{d}y$$

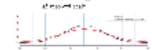
In general $m_{\mathsf{A}}(a) \ge \frac{1}{m_{\mathsf{H}}(a)}$!

Multi-Scale FEM

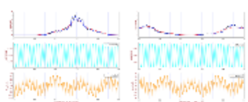
Method – Idea

Mathematics

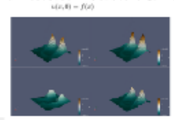
Model Problem



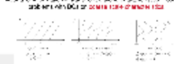
SLMsR Result



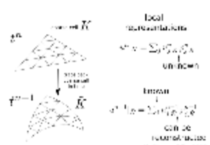
SLMsR 2D



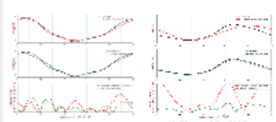
Naïve Approach



Reconstruction



Multi-Scale FEM
Upscaling Result



Features

Works, but:
- assumptions to flow field necessary
- no direct generalization to 2D/3D
- needs scale separation
- conservation not guaranteed

Remedy:
semi-Lagrangian reconstruction approach

Method – Idea

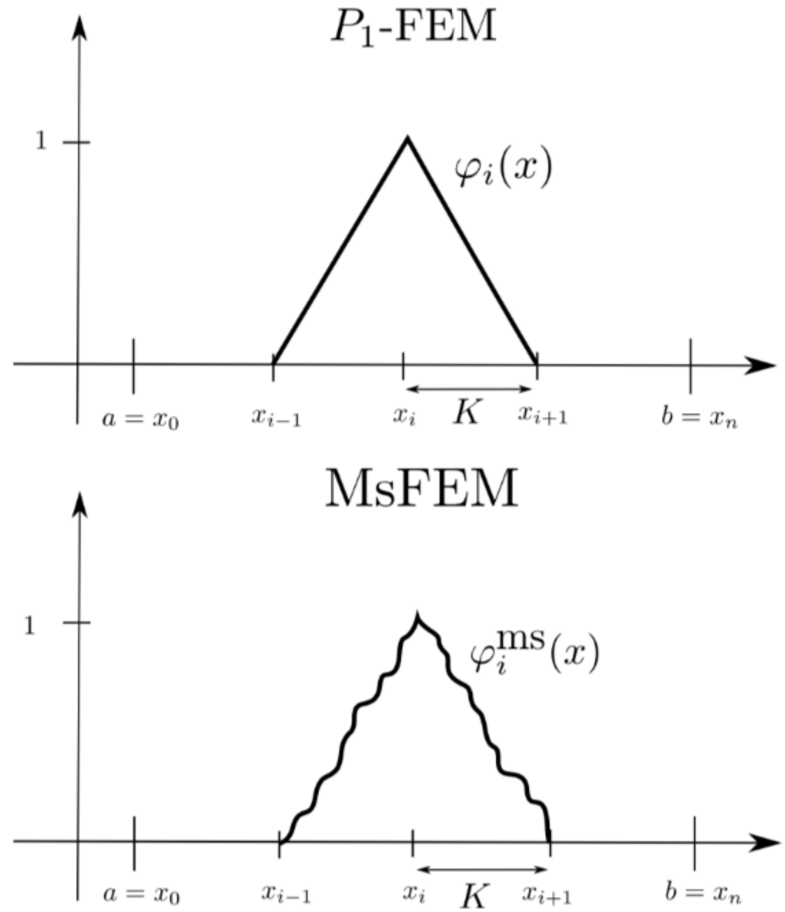
<u>MsFEM</u>

Model:

$$-\nabla \cdot (A^\varepsilon \nabla u^\varepsilon) = f$$

Idea: To capture the asymptotic structure of the solution modify the basis

$$-\nabla \cdot (A^\varepsilon \nabla \varphi_i^{\mathrm{ms}}) = 0 \quad \text{in } K$$
$$\varphi_i^{\mathrm{ms}}|_{\partial K} = \varphi_i|_{\partial K}$$



Mathematics

Theorem (Hou/Wu, 1999)

Let $u^\varepsilon \in H^2(\Omega)$ *solve the model problem and* $u^{\varepsilon,h} \in P^h$ *be the MsFEM solution.*
Then if $H < \varepsilon$

$$\left\|u^\varepsilon - u^{\varepsilon,h}\right\|_{H^1} \leq CH(|u^\varepsilon|_{H^2} + \|f\|_{L^2}) \, .$$

If $H > \varepsilon$ *and* $u^0 \in H^2 \cap W^{1,\infty}$ *is the solution to the homogenized problem then*

$$\left\|u^\varepsilon - u^{\varepsilon,h}\right\|_{H^1} \leq C(H + \varepsilon) \|f\|_{L^2} + C \left(\frac{\varepsilon}{H}\right)^{1/2} \left\|u^0\right\|_{W^{1,\infty}} \, .$$

Model Problem

Advection-Diffusion Eq.

$$\begin{aligned} \partial_t u(x,t) + c(x,t)\partial_x u(x,t) &= \partial_x(\mu(x,t)\partial_x u(x,t)) + g(x)\ , \quad && (x,t) \in I \times (0,T] \\ u(0,t) &= u(1,t)\ , && t \in (0,T] \\ u(x,0) &= f(x)\ , && x \in I \end{aligned}$$

$\mu = \mu(\frac{x}{\epsilon}, t)$ diffusion

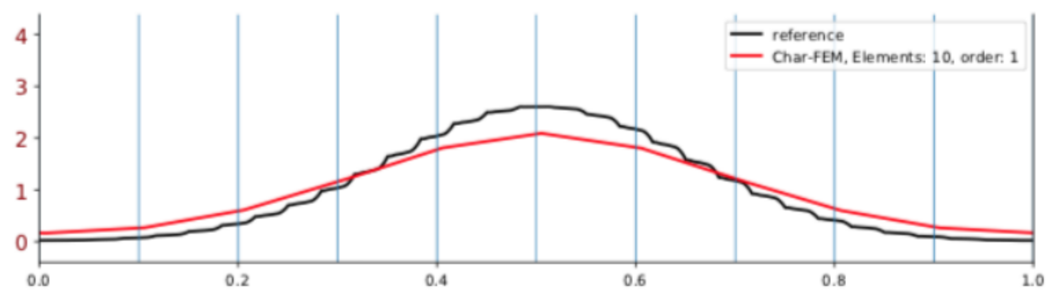
$c = c(x, \frac{x}{\delta}, t)$ wind

Assumptions to the data

$\epsilon \ll H$

$\delta \ll H$ or $\delta \gtrsim H$

H resolved scale



Naïve Approach

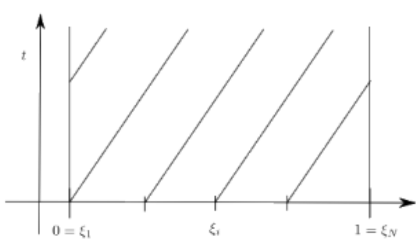
$$M_{ij}(t)\frac{\mathrm{d}}{\mathrm{d}t}u_j^H(t) + N_{ij}(\varphi_i^{\mathsf{ms}}, \dot{\varphi}_j^{\mathsf{ms}})(t)u_j^H(t) + C_{ij}(t)u_j^H(t) = D_{ij}(t)u_j^H(t)$$

where $u^H(x,t) = \sum_j u_j^H(t)\varphi_j^{\mathsf{ms}}(x,t)$.

$\rightarrow$ Does not work since the domain is badly decomposed

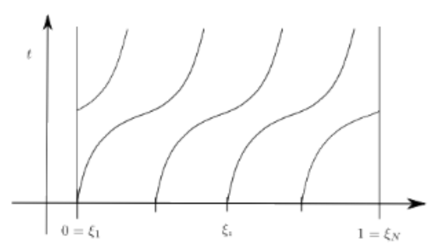
Remedy

Majority of information in transported along characteristics: Pose local problems with BCs on coarse scale characteristics.



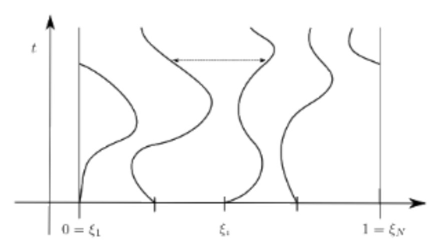
$c(x,t) \equiv \mathrm{const}$

$x(\xi,\tau) = \xi + c\tau$
$t(\xi,\tau) = \tau$



$c = c(t)$ or $c = c(x,t)$

$x(\xi,\tau) = \xi + \int_0^\tau \langle c\rangle(s)\mathrm{d}s$
$t(\xi,\tau) = \tau$

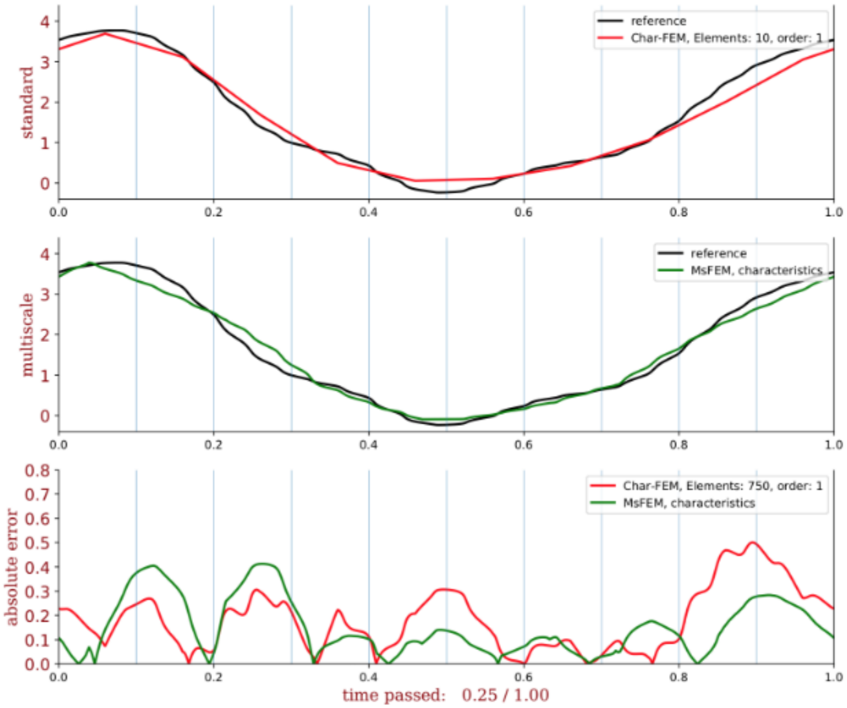


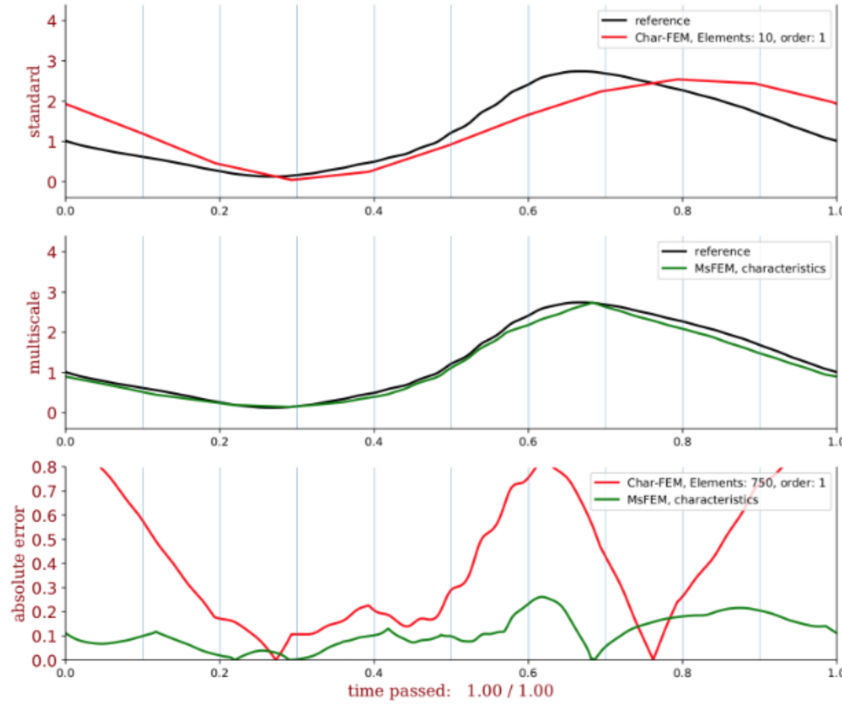
$c = c(x,t)$

$\frac{\mathrm{d}}{\mathrm{d}\tau}x(\xi,\tau) = c(x(\xi,\tau),\tau)$
$\frac{\mathrm{d}}{\mathrm{d}\tau}\,t(\xi,\tau) = 1$

Multi-Scale FEM

Upscaling Result





K. Simon, J.B., 2018

Features

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Remedy:

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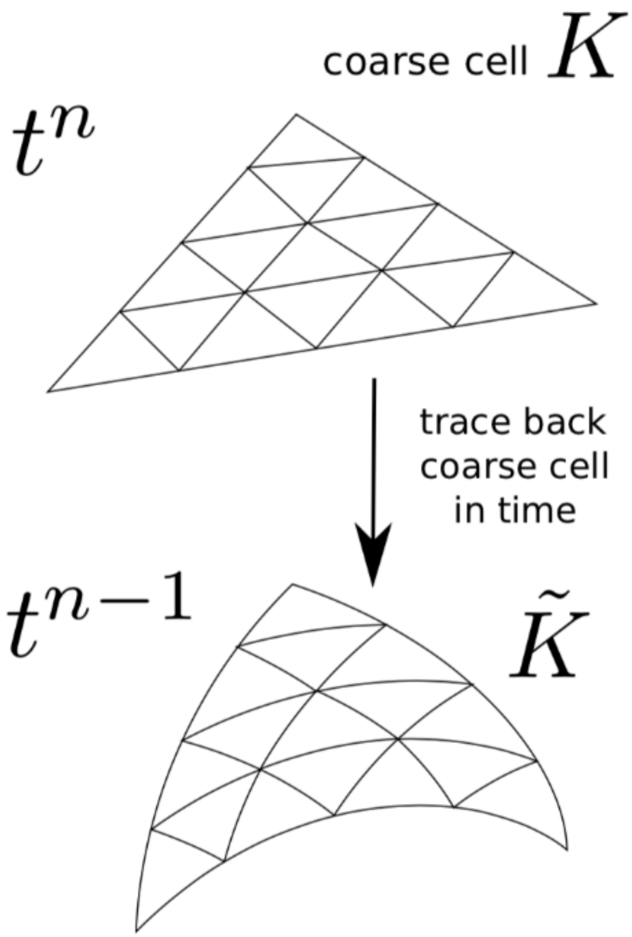
Reconstruction



local
representations:

$$u^n|_K = \sum_j u^n_{j,K} \varphi^n_{j,K}$$

unknown

known

$$u^{n-1}|_{\tilde{K}} = \sum_j u^{n-1}_{j,\tilde{K}} \varphi^{n-1}_{j,\tilde{K}}$$

can be
reconstructed

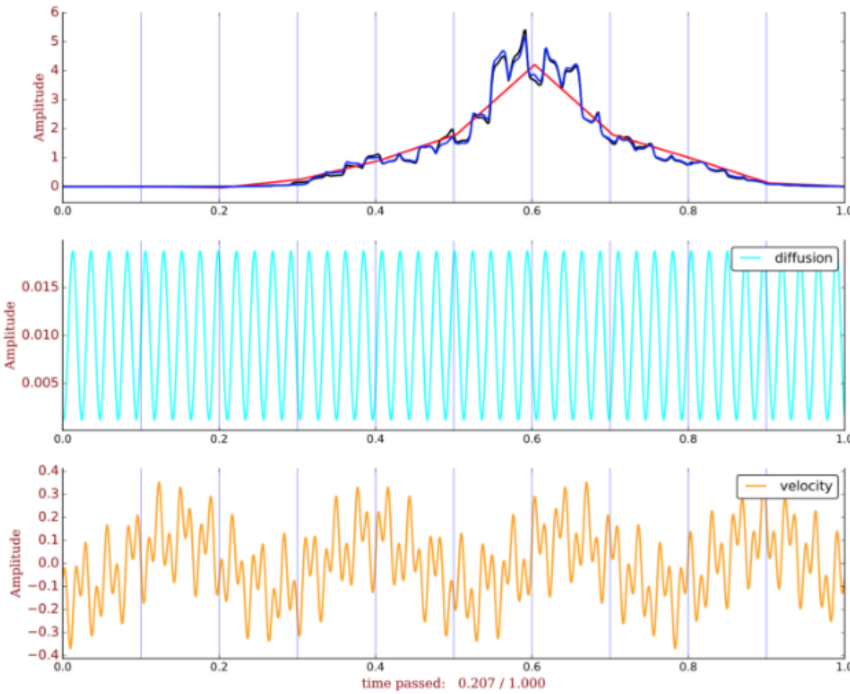
K. Simon, J.B., 2019

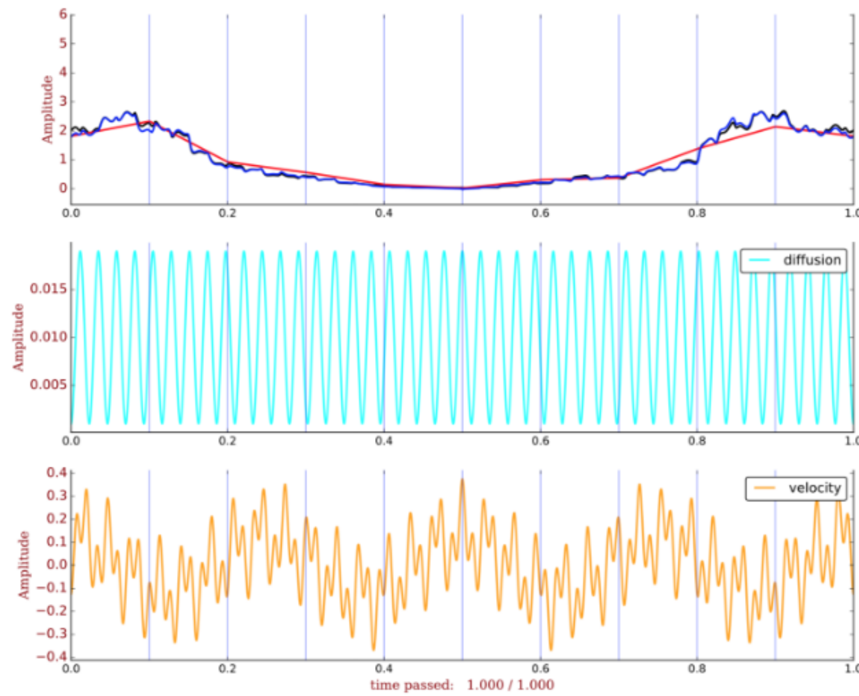
1D Example

$$u_t + [c(x, x/\delta, t)u]_x = (\mu(x, x/\varepsilon, t)u_x)_x \quad x \in (0,1)$$
$$u(0,t) = u(1,t)$$
$$u(x,0) = f(x)$$

$$c(x,t) = \frac{1}{2} + \frac{1}{8}(\cos(8\pi x) + \cos(62\pi x) + \cos(150\pi x))$$
$$\mu(x,t) = \cos(10\pi t)(0.01 + 0.009\cos(86\pi x))$$
$$n_{\mathrm{coarse}} = 10$$
$$n_{\mathrm{fine}} = 60$$
$$T_{\max} = 1$$
$$\delta t = 1/1000$$

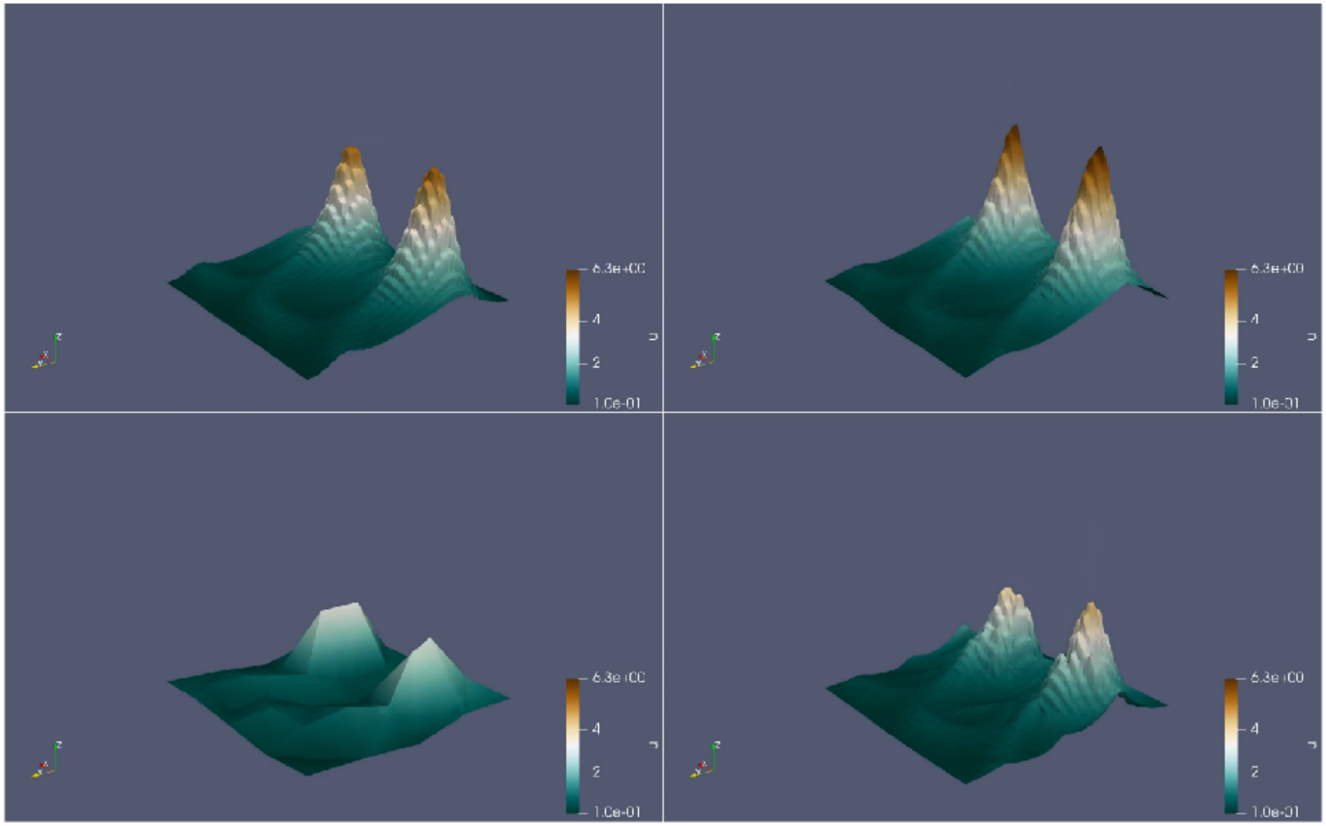
SLMsR Result



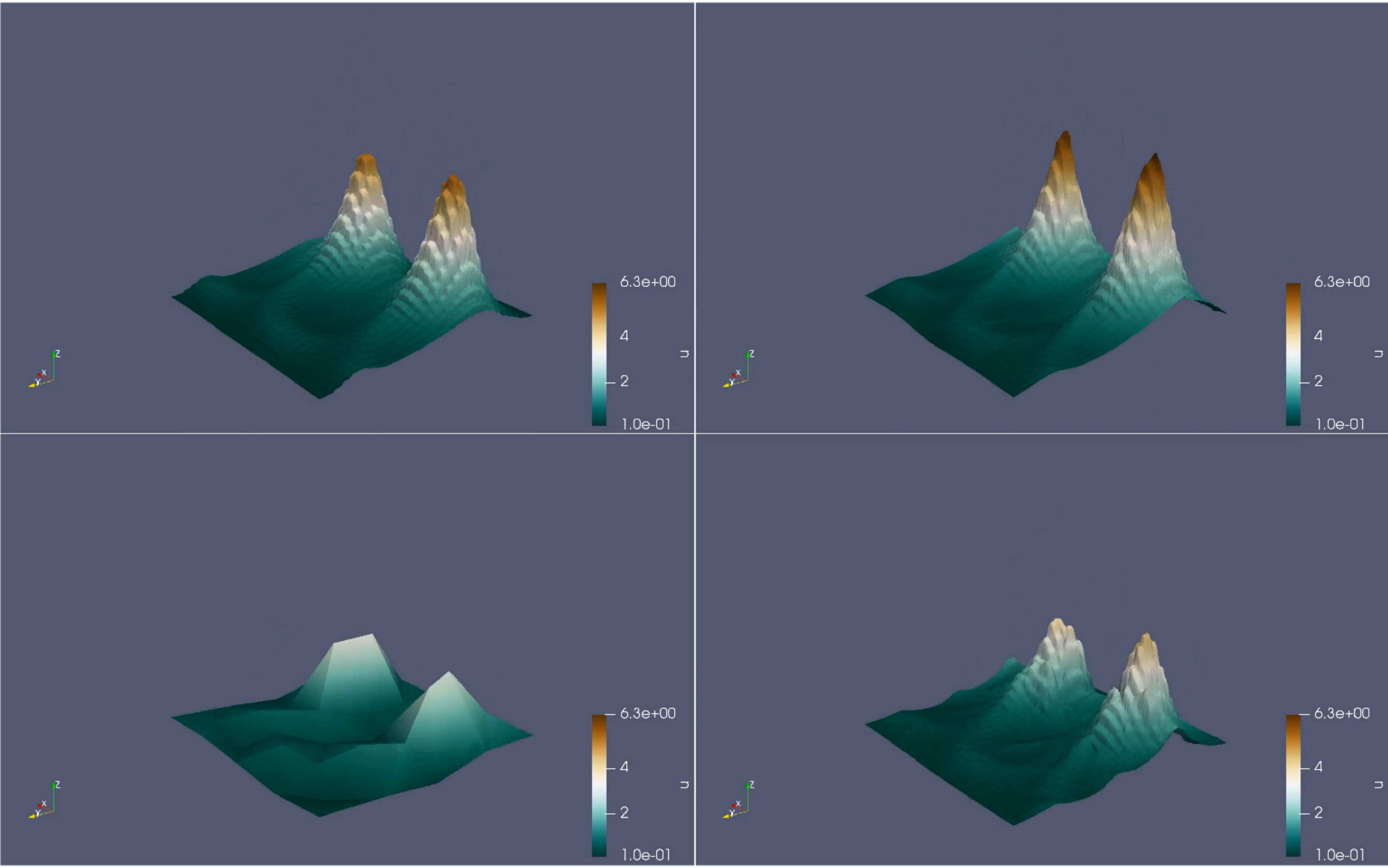


SLMsR 2D

$$u_t + \nabla \cdot (c(x, x/\delta, t)u) = \nabla \cdot (A(x, x/\varepsilon, t)\nabla u) + g \, , x \in \mathbb{T}_2$$
$$u(x, 0) = f(x)$$



$$u(x,0) = f(x)$$



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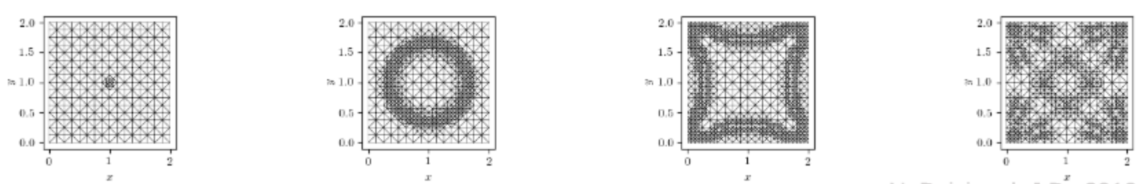
Conclusions

Downscaling:

- Adaptive Mesh refinement
- Efficiency
- Accuracy and Conservation

Upscaling:

- Homogeniztion viable approach (sometimes)
- Multiscale FEM in Lagrangian frame
- Multiscale reconstruction method



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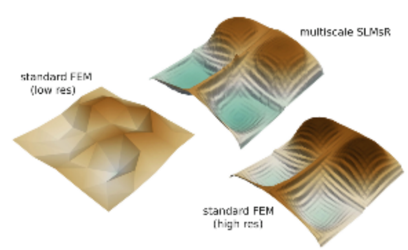

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Homogenization

Example

Homogenization Idea

Two Aspects of Multiscale Methods

Adaptive Meshes
Down-Scaling

Subgrid-scale Representation
Up-Scaling

Multi-Scale FEM

Method – Idea

Mathematics

Model Problem

SLMsR Result

SLMsR 2D

Naive Approach

Reconstruction

Features

Multi-Scale FEM
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Motivation

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Adaptive Mesh Refinement

Algorithm

Requirements

Efficient Data Structures

Tsunami wave propagation

Properties
- Mass/Momentum conserving
- Well Balanced
- Adaptive
- Wetting/drying