Analytical solutions for the free surface Euler & Navier-Stokes systems

Some ingredients for the validation of numerical schemes

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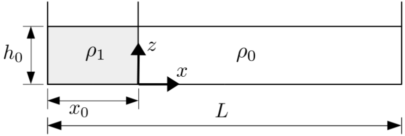
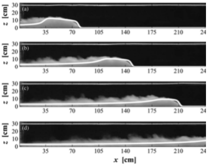
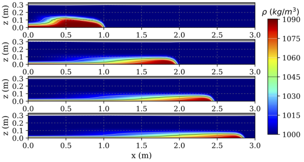
A typical example

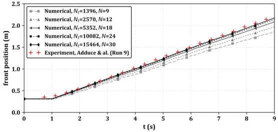
- The experimental device [Adduce *et al.*, J. Hydraul. Eng. 2012] (anim) (anim_low)



- Measurements versus simulation results

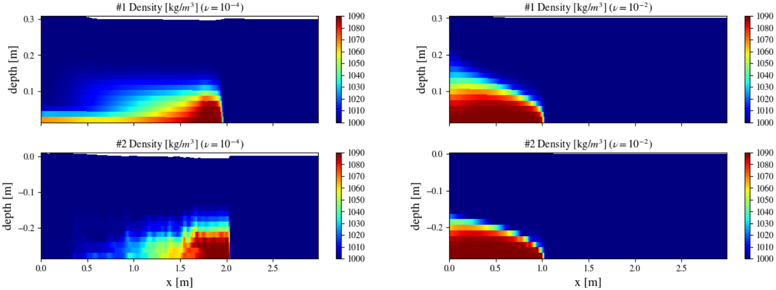




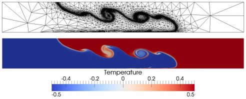


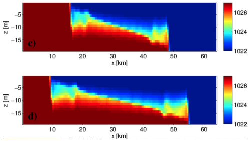
A typical example (cont'd)

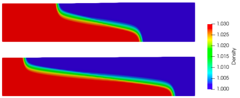
Comparison of 2 different codes (one of the experiments proposed by Adduce)



Other codes







See also Wroniszewski et al. Benchmarking of Navier-Stokes codes for free surface simulations, Coastal Eng., 91:1-17,2014

Conclusion

- The experiment
  - errors/uncertainties (initial conditions)
  - non-hydrostatic / hydrostatic
  - turbulence...
- The model

$$\begin{cases} \dot{\rho} + \mathrm{div}\,(\rho\underline{\mathbf{u}}) = 0 \\ \dot{\overline{\rho\underline{\mathbf{u}}}} + (\underline{\mathbf{u}}.\nabla)(\rho\underline{\mathbf{u}}) + \nabla p = \rho\mathbf{G} \\ \dot{\overline{\rho T}} + \mathrm{div}\,(\rho T\underline{\mathbf{u}}) = \mu\Delta T \\ \rho = \rho(T) \end{cases}$$

- The numerical codes
  - important differences
  - origins ?
- Analysis
  - no existence result, out of reach
  - shape of the interface ?
- Numerical analysis required
  - consistency, domain invariant, order of the scheme, discrete entropy
  - the two num. schemes are not consistent
  - accuracy versus stability
- Use of “reference” solutions

Contents

The analytical solutions

- Stationary / Non-stationary
- Euler / Navier-Stokes
- Hydrostatic / Non-hydrostatic
- Constant density / Variable density

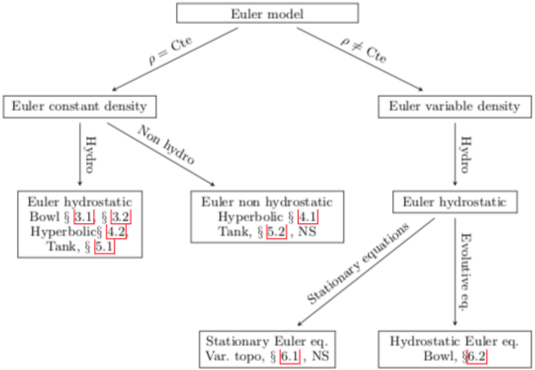
Allow to test

- Time discretization and/or space discretization
- Wet/dry interfaces
- Treatment boundary conditions
- Viscous terms discretization

Main idea

- Useful to have "reference" solutions when the numerical analysis is very complex
- Accuracy versus stability

The proposed analytical solutions



- Fully described in "Some analytical solutions for validation of free surface flow computational codes" `https://hal.archives-ouvertes.fr/hal-01831622`
- New solutions or extensions of exsting ones
- Also wave propagation

Shallow water analytical sol. of the Euler system

The 2d Euler system (u, w)

$$
\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\\
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = 0\\
&\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} = -g\\
&\frac{\partial \eta}{\partial t} + u_s\frac{\partial \eta}{\partial x} - w_s = 0, \quad u_b\frac{\partial z_b}{\partial x} - w_b = 0, \quad p|_s = p_0^a
\end{aligned}
$$

The SW assumption $u(x,z,t) = \bar{u}(x,t)$

$$
w = -(z - z_b)\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial z_b}{\partial x}
$$

Expression of the pressure

$$
-z\left(\frac{\partial^2 \bar{u}}{\partial x \partial t} + \bar{u}\frac{\partial^2 \bar{u}}{\partial x^2} - \left(\frac{\partial \bar{u}}{\partial x}\right)^2\right) + \frac{\partial^2 (z_b\bar{u})}{\partial x \partial t} + \bar{u}\frac{\partial^2 (z_b\bar{u})}{\partial x^2} - \frac{\partial \bar{u}}{\partial x}\frac{\partial (z_b\bar{u})}{\partial x} + \frac{\partial p}{\partial z} = -g
$$

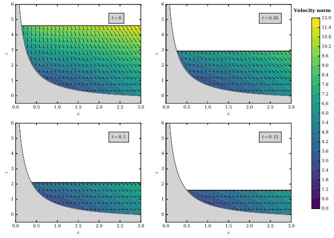
Hence
$$
\frac{\partial^2 p}{\partial x \partial z} = \frac{\partial^3 p}{\partial z^3} = 0 \quad \Rightarrow \quad p = p_0^a + \frac{a(t)}{2}(\eta^2 - z^2) + b(t)(\eta - z)
$$

Shallow water analytical sol. of the Euler system

The only SW solutions of the 3d Euler system

$$
\begin{aligned}
h(t,x,y) &= \max\left\{0, \alpha f(t) - b_0 - zb(x,y)\right\}\\
u(t,x,y,z) &= f(t)(x\cos\theta + y\sin\theta + \beta)\cos\theta\\
v(t,x,y,z) &= f(t)(x\cos\theta + y\sin\theta + \beta)\sin\theta\\
w(t,x,y,z) &= -f(t)(z+b_0)\\
p(t,x,y,z) &= p^{a,0}(t) + f^2(t)(\eta(x,y,t) - z^2) + \left(2b_0 f^2(t) + g\right)(\eta - z)
\end{aligned}
$$

where $f(t) = 1/(t - t_0 + t_1)$, $z_b(x,y) = \frac{c_0}{x\cos\theta + y\sin\theta + \beta} - b_0$, and $p^{a,0}(t)$ a given function.



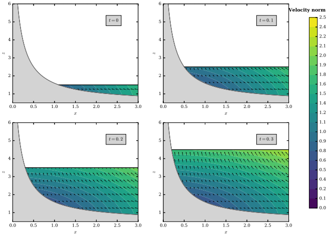
Animations
(With topography)
(Without topography)
also valid for Navier-Stokes

Shallow water sol. of the hydro. Euler system

The only SW sol. of the 3d hydrostatic Euler system

$$
\begin{aligned}
h(t,x,y) &= \max\{0, \alpha f(t) - b_0 - zb(x,y)\}\\
u(t,x,y,z) &= -f(t)(x\cos\theta + y\sin\theta + \beta)\cos\theta\\
v(t,x,y,z) &= -f(t)(x\cos\theta + y\sin\theta + \beta)\sin\theta\\
w(t,x,y,z) &= f(t)(z + b_0)\\
p(t,x,y,z) &= p^{a,0}(t) + p^{a,1}(x,t) + g(\eta - z)
\end{aligned}
$$

where $f(t) = 1/(t - t_0 + t_1)$, $z_b(x,y) = \frac{c_0}{x\cos\theta + y\sin\theta + \beta} - b_0$, and $p^{a,0}(t)$ a given function.

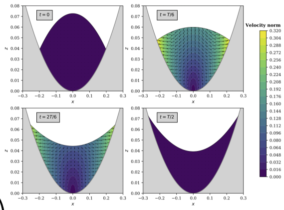


The hydrostatic & incompressible 3d Euler system

- The governing equations (hydrostatic)

$$\begin{cases} \nabla.\mathbf{U} = 0 \\ \rho_0 \left(\frac{\partial \mathbf{U}}{\partial t} + \nabla.(\mathbf{U} \otimes \mathbf{U})\right) + \nabla p = -\rho_0 \mathbf{g} \\ \frac{\partial \eta}{\partial t} + u_s \frac{\partial \eta}{\partial x} - w_s = 0, \quad u_b \frac{\partial z_b}{\partial x} - w_b = 0 \\ p|_s = p_0^a \end{cases}$$

- Analytical solution (anim)



$$H(t,x,y) = \max\left\{0, \frac{1}{r^2} f\left(\frac{r^2}{\gamma\cos(\omega t) - 1}\right)\right\}$$

$$u(t,x,y,z) = x\left(\beta\left(z - z_b - \frac{H}{2}\right) + \frac{\omega\gamma\sin(\omega t)}{2(1-\gamma\cos(\omega t))}\right)$$

$$v(t,x,y,z) = y\left(\beta\left(z - z_b - \frac{H}{2}\right) + \frac{\omega\gamma\sin(\omega t)}{2(1-\gamma\cos(\omega t))}\right)$$

$$w(t,x,y,z) = -\frac{\partial}{\partial x}\int_{z_b}^{z} u dz - \frac{\partial}{\partial y}\int_{z_b}^{z} v dz$$

$$p(t,x,y,z) = p^{a,0} + g(H + z_b - z)$$

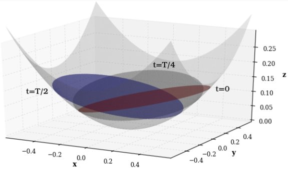
- Also moving bottom

(3d extension of Thacker)

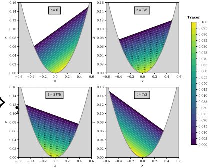
The 3d Euler system with varying density

- Hydrostatic Euler system (Boussinesq)

$$\begin{cases} \nabla.\mathbf{U} = 0 \\ \dfrac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{U}) = 0 \\ \rho_0 \left(\dfrac{\partial \tilde{\mathbf{U}}}{\partial t} + \nabla.(\tilde{\mathbf{U}} \otimes \mathbf{U}) \right) + \nabla p = \rho \mathbf{g} \\ \dfrac{\partial \eta}{\partial t} + u_s \dfrac{\partial \eta}{\partial x} - w_s = 0, \qquad u_b \dfrac{\partial z_b}{\partial x} - w_b = 0 \\ p\big|_s = p_0^a \end{cases}$$



- Analytical solution (anim)



$$H(t,x,y) = \max \left\{ 0, h_0 - \alpha \frac{(x - \eta \cos(\omega t))^2 + (y - \eta \sin(\omega t))^2}{2} \right\}$$

$$u(t,x,y,z) = -\eta\omega \sin(\omega t),$$

$$v(t,x,y,z) = \eta\omega \cos(\omega t),$$

$$w(t,x,y,z) = -\alpha\eta\omega \left(x \sin(\omega t) - y \cos(\omega t) \right),$$

$$p(t,x,y,z) = p^a(t) + \int_z^{H+z_b} \rho(\phi(t,x,y,z_1)) dz_1,$$

$$\phi(t,x,y,z) = a \left(H + z_b - z \right),$$

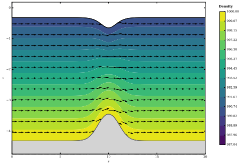
- For any nonnegative function $\rho : s \mapsto \rho(s)$

Remarks:

- ◇ ρ can be discontinuous (slow convergence of the num. scheme)
- ◇ Numerical stability of contact discontinuities?

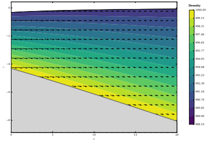
The 3d Euler system with varying density

- Stationary hydrostatic Euler system (Boussinesq)



$$\begin{cases} \nabla.\mathbf{U} = 0 \\ \nabla.(\rho\mathbf{U}) = 0 \\ \rho_0 \nabla.(\tilde{\mathbf{U}} \otimes \mathbf{U}) + \nabla p = \rho \mathbf{g} \\ u_s \frac{\partial \eta}{\partial x} - w_s = 0, \quad u_b \frac{\partial z_b}{\partial x} - w_b = 0 \\ p|_s = p_0^a \end{cases}$$

- Analytical solution



$$u(x,z) = f_1(x) + \frac{g\alpha}{2\rho_0} z^2$$

$$w(x,z) = -f_1'(x)(z - z_b(x)) + f_1(x) z_b'(x) + \frac{\alpha g}{2\rho_0} z_b^2(x) z_b'(x)$$

$$\rho(x,z) = \rho_0 - \alpha f_1(x)(z - z_b(x)) - \frac{\alpha^2 g}{6\rho_0}(z^3 - z_b^3(x))$$

with the function $f_1(x)$ defined by

$$f_1(x) = \frac{\alpha g}{2\rho_0}(h^2(x) - z_b^2(x)) - \frac{1}{\sqrt{3}\rho_0}\sqrt{(\alpha^2 g^2 h^3(x) - 6\rho_0^2 g)(h(x) + z_b(x)) - 6C_0}$$

A special function

- The LambertW function

$$\mathcal{LambertW}(x)e^{\mathcal{LambertW}(x)} = x$$

- In other words $\mathcal{LambertW}\big(\cos(x)\big)$ is the function $y = y(x)$ defined by

$$ye^y = \cos(x)$$

- The equation

$$ye^y = a\cos(x)$$

  admits several complex solutions, a unique real solution when $a \leq \frac{1}{e}$

- For small a

$$\mathcal{LambertW}\big(a\cos(x)\big) = a\cos(x) + \mathcal{O}(a^2)$$

Analytical solutions: linearized Euler

Proposition (with an energy balance)

Let f the function defined by

$$f(x) = -\frac{n}{k}\mathcal{L}ambert\mathcal{W}\Big(a\cos\big(\frac{x}{n}+b\big)\Big)$$

where $|a| < 1/e$, b, $k > 0$ and $n > 0$ are constants.
Up to terms in $\mathcal{O}(e^{-\frac{H_0 k}{n}})$, the functions H, u, w and p defined by

$$\begin{aligned}
H &= H_0 + f(kx - \omega t)\\
u &= \frac{\omega}{n}e^{\frac{k}{n}(z-H)}f(kx-\omega t)\\
w &= -\frac{n}{k}\frac{\partial u}{\partial x}\\
p &= p^a(t) + g(H_0 - z) - \frac{gan}{k}e^{\frac{k}{n}(z-H_0)}\cos\Big(\frac{kx-\omega t}{n}+b\Big)
\end{aligned}$$

are analytical solutions of the linearized Euler system iff $\frac{\omega}{k} = \sqrt{\frac{gn}{k}}$.

Analytical solutions: Euler

Proposition

Let f the function defined by

$$f(x) = -\frac{n}{k}\mathcal{LambertW}\Big(a\cos\big(\frac{x}{n}+b\big)\Big)$$

where $|a| < 1/e$, b, $k > 0$ and $n > 0$ are constants.
Up to terms in $\mathcal{O}(e^{-\frac{H_0 k}{n}})$, the functions H, u, w and p defined by

$$H = H_0 + f(kx-\omega t), \quad u = \frac{\omega}{n}e^{\frac{k}{n}(z-H)}f(kx-\omega t), \quad w = -\frac{n}{k}\frac{\partial u}{\partial x}$$

$$p = p^a(x,t) + g(H_0 - z) - \frac{ga^2 n}{2k}e^{\frac{2k}{n}(z-H_0)} - \frac{gan}{k}e^{\frac{k}{n}(z-H_0)}\cos\Big(\frac{kx-\omega t}{n}+b\Big)$$

$$p^a(x,t) = -\frac{ga^2 n}{2k}e^{\frac{2k}{n}(H-H_0)}$$

are analytical solutions of the Euler system iff $\frac{\omega}{k} = \sqrt{\frac{gn}{k}}$.

Standing waves

- Basic idea
$$\cos\left(\frac{kx-\omega t}{n}\right)+\cos\left(\frac{kx+\omega t}{n}\right)=2\cos\left(\frac{2kx}{n}\right)\cos\left(\frac{2\omega t}{n}\right)$$
- (anim)

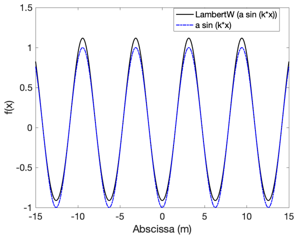
Proposition

With $f(x)=-\frac{n}{k}\mathcal{LambertW}\left(a\cos\left(\frac{x}{n}\right)\right)$, up to terms in $\mathcal{O}\left(e^{-\frac{H_0 k}{n}},a^3\right)$, the functions

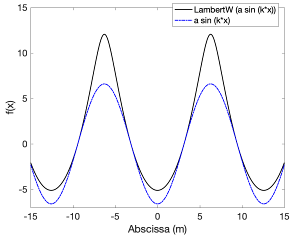
$$H=H_0+f(kx-\omega t)+f(kx+\omega t)+2\frac{a^2 n}{k}\cos^2\left(\frac{kx}{n}\right)$$
$$u=\frac{\omega}{n}e^{\frac{k}{n}(z-H)}\left(f(kx-\omega t)+f(kx+\omega t)++2\frac{a^2 n}{k}\cos^2\left(\frac{kx}{n}\right)\right)$$
$$\dots$$

are analytical solutions of the Euler system, linearized Euler system

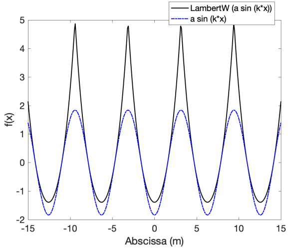
$$f(x) = -\frac{n}{k}\,\mathcal{L}ambert\mathcal{W}\left(a\cos\left(\frac{x}{n}+b\right)\right)$$



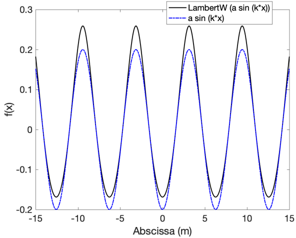
$a = 0.1$ m, $k = 0.1$ m^{-1}, $n = 1$



$a = 0.33$ m, $k = 0.1$ m^{-1}, $n = 2$



$a = e^{-1}$ m, $k = 0.2$ m^{-1}, $n = 1$



$a = 0.2$ m, $k = 1$ m^{-1}, $n = 1$

Conclusion: about these analytical solutions

- Accessible by simulations prescribing
  - geometrical domain
  - initial and boundary conditions
- Stability
  - some of them also valid for Navier-Stokes (viscous fluid)
  - but not enough, analysis result required
  - sometimes convergence very low (variable density case)
- Should be completed with "reference solutions"